GR Questions — HT 2017

Part A.

1. Estimate the width of the hydrogen line (5.9 μeV) that we would see if we observed the spectrum of gas orbiting a black hole.

(Assume circular orbits, minimum possible J.)

2. You sit at fixed r, θ, ϕ outside a black hole (Schwarzschild) and throw a ball radially towards the horizon.

How long does it take to watch it fall in?

3. Do covariant derivatives commute on scalars?

4. Propose some generalisations of the Klein-Gordon equation to curved space.

5. Consider

$$T_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} (\nabla^c \phi \nabla_c \phi + m^2 \phi^2).$$

When is $\nabla^a T_{ab} = 0$ satisfied?

In the equatorial plane, the Kerr black hole is

$$g = -A\,dt^2 + B(d\phi - \Omega\,dt)^2 + C\,dr^2$$

where A, B, Ω, C are functions of r:

$$A = \frac{\Delta r^2}{\Sigma^2}, \quad B = \frac{\Sigma^2}{r^2}, \quad C = \frac{r^2}{\Delta}, \quad \Omega = \frac{r_s r a}{\Sigma^2}$$

where

$$\Delta = r^2 - 2Mr + a^2 \qquad \Sigma^2 = (r^2+a^2)^2 - a^2\Delta$$

$$r_s = 2M$$

and M, a are parameters.

6. For Kerr as above, $g^{rr} = 0$ for what values of r?

7. Consider $K = \partial_t + \Omega(r)\partial_\phi$. For what values of r is K null for this metric?

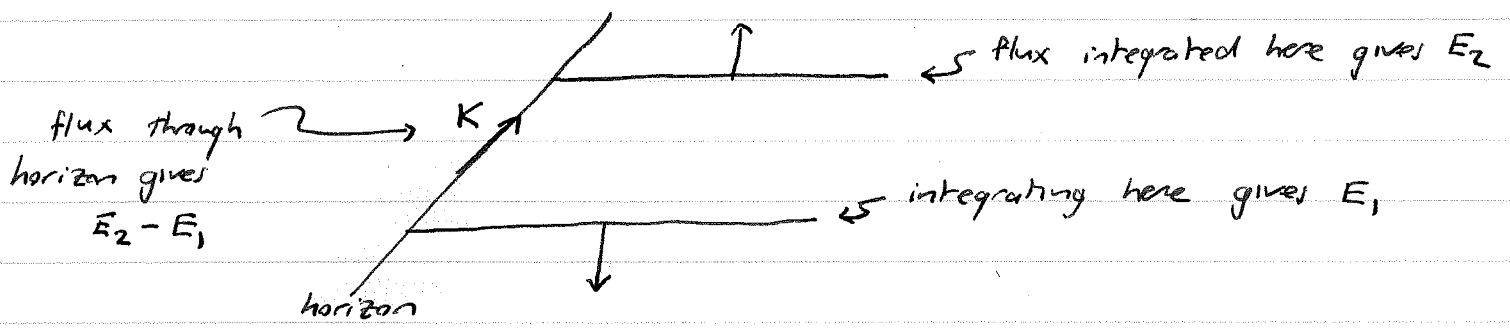
8. Let $\chi = \partial_t$. Compute $\chi \cdot K$.

9. $J_a = T_{ab}\chi^b$ is a conserved current whose integral corresponds to energy. At the horizons, where $g^{rr} = 0$, we can regard K as a normal vector. For T_{ab} as in Q5 evaluate the flux

$$\mathcal{E} = T_{ab} K^a \chi^b$$

at a horizon for $\Phi = c\, e^{-iEt} e^{iJ\phi}$.

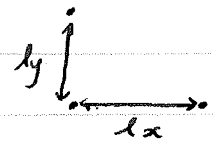
For what E, J is the flux positive?



Part B.

10. Consider a perturbation $\eta + h$ such that $h_{00} = h_{0i} = 0$ $(i = 1 \ldots 3)$. Find the geodesic equation to first order in h and conclude that if $U^i(t=0) = 0$, the geodesics will be straight lines of constant (x, y, z).

11. Consider three such geodesics at $(0,0,0)$, $(s,0,0)$, $(0,s,0)$. What are the proper separations of these to first order in h?



How are these separations affected by a gravitational wave? Show this.

12. Suppose two (Schwarzschild) black holes (mass M_1, M_2) collide to form one (mass M). Hawking (1971) says the total horizon area must increase. Obtain an upper bound on the energy of gravitational radiation released.

Recall the geodesic deviation equation

$$\frac{D^2 N^a}{D\tau^2} = -R^a{}_{bcd}\, U^b U^c N^d$$

where N measures the displacement between neighbouring geodesics and $\frac{D}{D\tau} = U^a \nabla_a$ — for U the velocity.

13. If you want to, show that

$$R_{abcd} = \frac{1}{2}\left(h_{ac,bd} + h_{bd,ac} - h_{bc,ad} - h_{ad,bc}\right)$$

in the weak limit.

14. To leading order in v/c show that

$$\frac{D^2 N^i}{D\tau^2} = \frac{1}{2} h_{ij,00} N^j$$

in the weak limit.

15. For $u = (1,0,0,0)$ show

$$\nabla_u \nabla_u N^i = \ddot{N}^i + \frac{1}{2}\ddot{h}_{ij} N^j + \dot{h}_{ij} \dot{N}^j,$$

where dot is $\frac{\partial}{\partial t}$, to first order in h.

16. Conclude that the geodesic deviation equation gives

$$\ddot{N}^i = -\dot{h}_{ij} \dot{N}^j.$$

Compare with Q10.

17. Find R_{ab} and R in terms of h and conclude that the linearised field equation is

$$\partial_c \partial^c \bar{h}_{ab} = 16\pi G\, T_{ab}$$

if we adopt Lorenz gauge: $\partial^a \bar{h}_{ab} = 0$.

18. Using Fourier transforms, say, obtain the Green's functions for $\Box = \partial_c \partial^c$

$$G_\pm(t, \underline{x}) = \frac{1}{4\pi|\underline{x}|}\,\delta(t \pm |\underline{x}|).$$

19. Thus obtain

$$\bar{h}_{ab}(t, \underline{x}) = 4G \int d^3y \, \frac{1}{|\underline{x}-\underline{y}|} \, T_{ab}(t \mp |\underline{x}-\underline{y}|, \underline{y}).$$

20. For large $|\underline{x}| = r$ show that

$$\frac{1}{|\underline{x}-\underline{y}|} = \frac{1}{r}\left(1 + \frac{\underline{x}\cdot\underline{y}}{r} + O(1/r^2)\right)$$

21. By considering $(T^{ab} x^i x^j)_{,ab}$ observe that

$$2T^{ij} = \text{total divergence} + (T^{00} x^i x^j)_{,00}.$$

Infer the leading (in $1/r$) contribution

$$\bar{h}_{ij}(t, \underline{x}) = \frac{2G}{r} \cdot \ddot{I}_{ij}$$

where I is the 2nd moment of T^{00}.

22. Compute the frequency of the gravitational waves emitted by a binary with orbital frequency ω and similar masses.

23. Compute the power radiated.

For Q23 you may use $P = -\frac{G}{5}\langle \dddot{J}_{ij}\, \dddot{J}^{ij} \rangle$ where J_{ij} is $I_{ij} - \delta_{ij} I_{kk}/3$ (sum k).

24. State or prove the virial theorem and find the (Newtonian) total energy of the binary.

25. If the ~~stars~~ stars are separated initially by $2R_\odot$, how long until they collide?

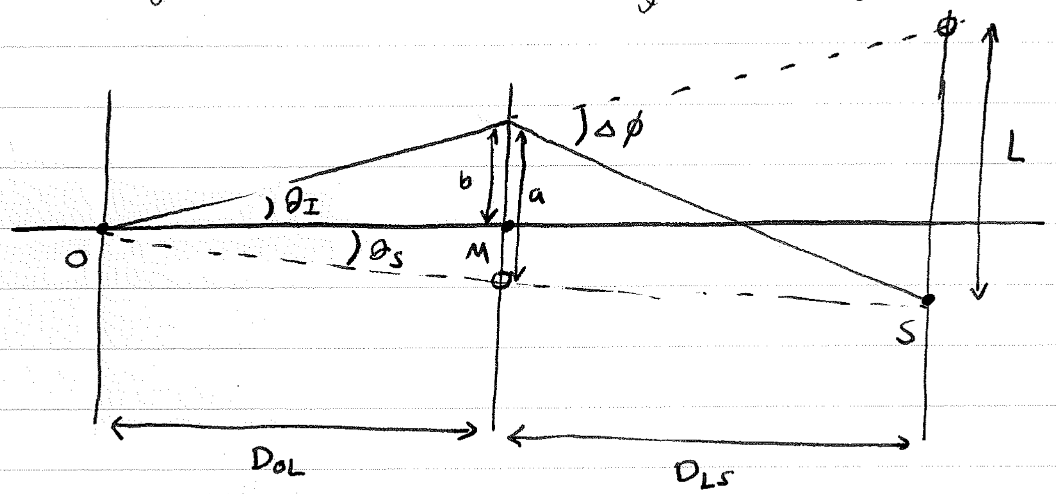
26. If you're game, perform a similar analysis for a small mass slingshotting a large one (using the Newtonian parabolic orbit).

The solution is in Peters 1969, Phys. Rev. D.

27. Derive or recall the deflection angle of light

$$\Delta\phi = \frac{4GM}{b}$$

caused by a point mass. By considering the geometry



show that the introduction of the mass M displaces the image on the lens plane by

$$a = \frac{D_{OL}\, D_{LS}}{D_{OS}}\, \Delta\phi$$

where $D_{OS} = D_{OL} + D_{LS}$.

28. Argue that an arbitrary density profile $\rho(\underline{x})$ on the lens plane will give displacements

$$\underline{a}(x) = \frac{D_{OL} D_{LS}}{D_{OS}} \cdot 4G \int \frac{\rho(\underline{y})\,(\underline{y}-\underline{x})}{|\underline{y}-\underline{x}|^2}\, d^2\underline{y}.$$

What do we need to know in order to infer the density $\rho(y)$ of the lens?

29. when would you see an Einstein ring? And how big would it be?

30. Consider the Euclidean metric

$$ds^2 = n(x)\, dx^i dx^i, \qquad n = 1 - 2\Phi$$

and integrate the Lagrange equation for

$$L = n \dot{x}^i \dot{x}^i$$

to find (with $|\dot{x}| = 1$) that

$$\dot{x}^i(t_f) - \dot{x}^i(t_i) = \int_{t_i}^{t_f} \nabla_\perp \ln n \; dt .$$

What is the small Φ limit? What have you calculated?

Part e.

31. If you want to, show that

$$g = dt^2 - a^2(t)\, \gamma_{ij}\, dx^i dx^j$$

has, for γ_{ij} one of the constant curvature metrics

$$R_{00} = -3\ddot{a}/a \qquad R_{ij} = (\ddot{a}a + 2\dot{a}^2 + 2k)\, \gamma_{ij} .$$

What is R? What is the curvature of the $t = \text{const}$ slices?

32. Infer Friedman's equations from Q31. One of them is:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} \qquad \star$$

where $\rho = \rho_M + \rho_\Lambda$, $\rho_\Lambda = \frac{1}{8\pi G}\Lambda$.

33. Write $(\star)$ in terms of H, H_0, a, $\Omega_M, \Omega_k, \Omega_\Lambda$, if the universe has only non-relativistic matter.

Motivate the definition of $\Omega_{M,k,\Lambda}$.

34. If $\Omega_k = \Omega_\Lambda = 0$, what is the age of the universe in terms of H_0?

35. In the expansion

$$\frac{a(t_0)}{a(t_1)} = 1 + H_0(t_0 - t_1) + \frac{1}{2} q_0 H_0^2 (t_0 - t_1)^2 + \cdots$$

what is q_0 in terms of a and its derivatives?

36. Use the 2nd Friedman equation to write q_0 in terms of $\Omega_M, \Omega_\Lambda, \Omega_k$ and comment on why this is interesting.

37. Recall

$$n \sim g\, T^{3/2}\, e^{-m/k_B T}$$

for a non-relativistic species of particles. Consider $p^+ + e^- \leftrightarrow H + \gamma$ and show

$$\frac{1-X}{X^2} \sim (n_p + n_H)\, T^{-3/2}\, e^{B/k_B T}$$

where $X = n_e / n_p + n_H$ and the system is assumed to be electrically neutral.

38. The following facts

- $B = 13.6$ eV
- $n_B / n_\gamma = 6 \times 10^{-10}$ today
- $n_{atoms} = 0.76\, n_B$ today

are sufficient to infer that $X = 0.1$ when T was a certain temperature.

When correct constants are included, this is found to be $T = 3{,}400$ K.

39. The CMB has temperature 2.75K today. How many times greater is $a(t_0)$ than $a(t_{rec})$, if t_{rec} is the time when X was 0.1?