The Remembrance of Things Past

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Abstract

We compute the memory effect at $\mathscr{I}^+$ and $\mathscr{I}^-$ for graviton scattering in two ways. First, using linearised gravity and an ansatz for the non-linear graviton contribution. Second, as a consequence of the soft factorisation theorem. We find that the effect is the same for $\mathscr{I}^+$ and $\mathscr{I}^-$, given the identification of null generators provided by Ashtekar's construction of i^0. This surprising result can be understood as a consequence of Strominger's proposed symmetry for graviton scattering. We also show that the memory effect for massive particle scattering can be derived from a soft factorisation. In both cases we comment on the evolution of the Bondi mass.

Contents

1 Introduction

The memory effect is the permanent displacement of a gravitational wave detector due to gravitational waves. It was first associated to astrophysical processes in the weak-field non-relativistic limit [1]. Assuming the radiation produced by these processes is long wavelength, one finds a net displacement which arises from changes in the 'mass aspect' of the objects. Later, Christodoulou [2] computed the memory effect in general for any asymptotically flat spacetime. He found that both changes in the mass aspect and the flux of massless radiation contribute to the effect. It was recently noticed that Christodoulou's result can be related to the soft factorisation of amplitudes originally found by Weinberg [3]. This connection was made after earlier work by Strominger and collaborators [4,5] revealed that Weinberg's soft factor is a Ward identity for a conjectured symmetry group of the scattering problem. In this report we study the memory effect for graviton scattering and its relationship to the soft factor. We begin by reviewing the conformal description of null infinity and derive how geodesic deviation near $\mathscr{I}$ is related to the shear of outgoing null geodesics. In §3 we discuss the symmetries of $\mathscr{I}$ and review the group of gauge-preserving symmetries known as the BMS group. We derive the graviton soft factor in §4 and explain why it is tree-level exact, commenting on its relationship to the BMS group. Then, in §5 we derive the known formula for graviton scattering memory in two ways: making clear the relationship with the soft factorisation. Finally, we show how the memory effect for massive particle scattering also follows from a soft factor.

2 Null Infinity

To study gravitational radiation we are interested in spacetimes which become flat at long distances from the sources of the radiation. Early work studied spacetimes whose curvature decayed sufficiently fast along outgoing null geodesics [6]. A more helpful definition was proposed by Penrose [7] who studied spacetimes that allow a conformal compactification. Recall that Minkowski spacetime can be conformally rescaled and seen as the interior of a manifold with boundary. In this case, the boundary is given by two null hypersurfaces $\mathscr{I}^{\pm}$ and three points $i^{\pm,0}$ (which are apexes of light cones). Following Penrose, we can define a class of spacetimes that have a similar asymptotic structure. We call (M, g_{ab}) *asymptotically flat* if a conformal rescaling of M is the interior of some manifold with boundary $(\hat{M}, \hat{g}_{ab})$ which satisfies: (i) $\hat{g}_{ab} = \Omega^2 g_{ab}$ on the interior of $\hat{M}$, for some non-negative Ω; (ii) both Ω and $\hat{g}_{ab}$ are C^k smooth on $\hat{M}$ for $k > 2$; (iii) $\Omega = 0$ and $\mathrm{d}\Omega \neq 0$ on the boundary of $\hat{M}$; (iv) Einstein's vacuum equations (with zero cosmological constant) hold in a neighbourhood of the boundary. We claim that the boundary is locally a shear-free null hypersurface. To establish this, recall that (in four dimensions) the Ricci tensor of g_{ab} is related to the Ricci tensor of $\hat{g}_{ab}$ by

$$R_{ab} = \hat{R}_{ab} + 2\Omega^{-1}\hat{\nabla}_a\hat{\nabla}_b\Omega + \hat{g}_{ab}\left(\Omega^{-1}\hat{\nabla}^2\Omega - 3\Omega^{-2}\hat{\nabla}^a\Omega\hat{\nabla}_a\Omega\right). \tag{1}$$

Taking the trace with $\hat{g}^{ab}$ gives

$$\Omega^2 R = \hat{R} + 6\Omega^{-1}\hat{\nabla}^2\Omega - 12\Omega^{-2}\hat{\nabla}^a\Omega\hat{\nabla}_a\Omega \tag{2}$$

By condition (iii), the boundary of $\hat{M}$ is given by $\Omega = 0$ and $\hat{N}_a = -\hat{\nabla}_a\Omega$ is a non-vanishing normal to the boundary[1]. By (ii), $\hat{R}_{ab}$ and $\hat{\nabla}^2\Omega$ are C^{k-2} smooth on and near the boundary. Moreover, $R = 0$ by condition (iv). So equation (2) implies

$$\hat{N}_a\hat{N}^a \approx 0,$$

where '$\approx$' means equality when $\Omega = 0$. Moreover, we decompose the Ricci tensor as $R_{ab} = \Phi_{ab} + g_{ab}R/4$ and use equation (1) to infer that

$$\Phi_{ab} = \hat{\Phi}_{ab} + 2\Omega^{-1}\hat{\nabla}_a\hat{\nabla}_b\Omega - \frac{1}{2}\Omega^{-1}\hat{\nabla}^2\Omega,$$

[1]Our conventions are that $\hat{N}^a \equiv \hat{g}^{ab}\hat{N}_b = N^b$ and $N_a = g_{ab}N^b = \Omega^{-2}\hat{N}_a$.

which implies

$$\hat{\nabla}_a \hat{N}_b \approx \frac{1}{4}\hat{g}_{ab}\hat{\nabla}_c\hat{N}^c.$$

So the boundary $\Omega = 0$ is locally a null, shear-free hypersurface. In ref. [7] Penrose adopts the condition that every null geodesic in M can be extended to have two intersections with the boundary of $\hat{M}$. Assuming that $\hat{M}$ is connected, this implies that the boundary has two connected components $\mathscr{I}^{\pm}$ each with the topology of $\mathbb{R} \times \mathbb{S}^2$ [7]. We call these null hypersurfaces *future* and *past null infinity* . In the compactification of Minkowski space there is a point i^0, spatial infinity, such that $\mathscr{I}^+ \cup \mathscr{I}^- \cup i^0$ is a null cone (with i^0 the apex). For Minkowski space this allows us to identify null geodesics in $\mathscr{I}^+$ with those in $\mathscr{I}^-$. However, by our definition of asymptotic flatness, the point i^0 is not included[2] in $\hat{M}$. So, following Ashtekar and Hansen [8], we enlarge our definition of $\hat{M}$ to include a point i^0 such that $\mathscr{I}^+ \cup \mathscr{I}^- \cup i^0$ is a null cone.

2.1 The Asymptotic Behaviour of Fields

It is possible to deduce the asymptotic behaviour of conformal densities from their conformal weights. To make this precise we must describe the outgoing null geodesics orthogonal to $\mathscr{I}^+$; that is, those that intersect $\mathscr{I}^+$ at one point only. Suppose that these geodesics are generated by the flagpole $l^a = o^A\bar{o}^{\dot{A}}$ for some spin frame o^A, ι^A. We specialise to a frame that is parallel transported along these geodesics and introduce an affine parameter r normalised so that $l^a\nabla_a r = 1$. Then any scalar conformal density $\mathcal{A}$ of weight $-w$ admits an expansion

$$\mathcal{A} = r^{-w}\mathcal{A}_0 + r^{-w-1}\mathcal{A}_1 + ...,$$

where the $\mathcal{A}_i$ are scalars that are constant along the l^a geodesics. This can be generalised to the components of a conformal tensor density: one finds that a component with p o^A or $\bar{o}^{\dot{A}}$ indices has leading order behaviour $O(r^{-w-p})$. This is called the peeling theorem: see appendix A for the proof. Using this result, it is possible to solve Einstein's equations

[2]Minkowski space is asymptotically flat. In null coordinates $u = \tan U$ and $v = \tan V$ one has $g = -\mathrm{d}u\mathrm{d}v = -\Omega^{-2}\mathrm{d}U\mathrm{d}V$ where $\Omega = \cos U \cos V$. We can use U, V to describe the compactified spacetime with $\hat{g} = -\mathrm{d}U\mathrm{d}V$. We identify i^0 as $U = -\pi/2$ and $V = \pi/2$. But $\mathrm{d}\Omega$ vanishes at this point.

for a general asymptotically flat spacetime in a neighbourhood of $\mathscr{I}$ [9,10]. To do this, it is helpful to write the field equations in terms of the 16 spin-frame components of the connection ∇: which are called spin coefficients. We adopt the abbreviations $D = l \cdot \nabla$, $\delta = m \cdot \nabla$, $\delta' = \bar{m} \cdot \nabla$ and $D' = n \cdot \nabla$. We will use lower case greek letters for the spin coefficients of $\nabla_{B\dot{B}} o^A$ while the corresponding components of $\nabla_{B\dot{B}} \iota^A$ are given primed greek letters: our conventions match §4.5 of [11]. Many of the spin coefficients are conformal densities. For example consider $\sigma \equiv o^\alpha \delta o_\alpha$. The conformal transformation $g_{ab} \to \Omega^2 g_{ab}$ amounts to a rescaling of the spin frame according to $o^\alpha \to \Omega^{-1} o^\alpha$, and $\iota^\alpha \to \iota^\alpha$. Under this, σ transforms as

$$\sigma \ \mapsto \ \hat{\sigma} = \hat{o}^A \hat{o}^B \hat{\bar{\iota}}^{\dot{B}} \hat{\nabla}_{B\dot{B}} \hat{o}_A = \Omega^{-2}\sigma + \Omega^{-2} o^A o^B \bar{\iota}^{\dot{B}} o_B \nabla_{A\dot{B}} \log \Omega = \Omega^{-2}\sigma.$$

So σ has conformal weight -2 and has an asymptotic expansion $\sigma = \sigma^0 r^{-2} + O(r^{-3})$. A second important example is the Weyl tensor, C_{abcd}. This is the conformally invariant part of R_{abcd}. We write $C_{abcd} = \Psi_{ABCD}\epsilon_{\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}}$+c.c.. Notice that the symmetries of R_{abcd} imply that Ψ_{ABCD} has five independent components. So we define: $\Psi_0 = \Psi_{0000}$, $\Psi_1 = \Psi_{0001}$, $\Psi_2 = \Psi_{0011}$, $\Psi_3 = \Psi_{0111}$, $\Psi_4 = \Psi_{1111}$. Now, given the conditions for asymptotic flatness above, one can prove that $\Psi_{ABCD} \approx 0$ [12]. This means that we can define a smooth field ψ_{ABCD} with conformal weight -1 such that $\psi_{ABCD} = \Psi_{ABCD}$ for the metric g_{ab}. But then, by the peeling theorem for the components of ψ_{ABCD} we find

$$\Psi_k = \Psi_k^0 r^{-(5-k)} + \Psi_k^1 r^{-(6-k)} + O(r^{-(7-k)}).$$

Using expansions of this kind as an ansatz for the field equations one can solve for the functions that appear at each order. We return to this in the next section.

2.2 The Bondi Gauge

We have already specialised to a frame that is parallel transported with respect to the tetrad vector l and we have demanded that l is tangent to the outgoing null geodesics.

In this section we will make further gauge choices. The resulting frame is said to be in *Bondi gauge* because it allows for the introduction of the coordinates that were used by Bondi [6]. We impose two conditions. First, we demand that the outgoing null geodesics are tangent to null hypersurfaces. The intersections ('cuts') of these with $\mathscr{I}^+$ will then foliate $\mathscr{I}^+$. We impose this condition by demanding that $l^* = \mathrm{d}u$ for some u, where the $*$ denotes the dual with respect to g_{ab}. We call such a coordinate a 'Bondi parameter'. By the orthogonality of the tetrad, $D'u = 1$. So u is an increasing parameter on $\mathscr{I}^+$. Our second condition is that the metric induced on these $u =$ const. cuts is that of the unit sphere. In stereographic coordinates, this metric is $\mathrm{d}s^2 = 4P^{-2}\mathrm{d}z\mathrm{d}\bar{z} \equiv 2\gamma_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}$ where $P = 1 + z\bar{z}$. We now give the result of solving the field equations in Bondi gauge. Note that, in spin coefficients, the first gauge condition implies that ρ is real and $\tau = \bar{\alpha} + \beta$. Moreover, since our frame is parallel transported the spin coefficients involving D vanish: $\kappa = \epsilon = \gamma' = \tau' = 0$. The leading behaviour of the nonvanishing spin coefficients is then

$$\rho = -r^{-1} + O(r^{-3}) \qquad \alpha = -\beta' = \tfrac{1}{2}zr^{-1} + O(r^{-2}) \qquad \sigma' = -\dot{\bar{\sigma}}^0 r^{-1} + O(r^{-2}) \tag{3}$$

$$\sigma = \sigma^0 r^{-2} + O(r^{-4}) \qquad \beta = -\alpha' = -\tfrac{1}{2}\bar{z}r^{-2} + O(r^{-2}) \qquad \rho' = \rho'_0 r^{-1} + O(r^{-2}) \tag{4}$$

$$\tau = -\frac{1}{2}\Psi_1^0 r^{-3} + O(r^{-4}) \quad \gamma = -\epsilon' = -\tfrac{1}{2}\Psi_2^0 r^{-2} + O(r^{-3}) \quad \kappa' = \Psi_3^0 r^{-1} + O(r^{-2}). \tag{5}$$

Notice that $2\gamma = -l_a D' n^a \approx 0$. So, for a Bondi parameter u, $0 = D'^2 u \approx n^a n^b \nabla_a \nabla_b u$. The field equations also determine the leading behaviour of two Weyl tensor components[3]: $\Psi_3^0 = \eth\dot{\bar{\sigma}}^0$ and $\Psi_4^0 = -\ddot{\bar{\sigma}}^0$. Moreover, one can expand the Bianchi identity for the Weyl tensor ($\nabla_{A\dot{A}}\Psi^{ABCD} = 0$) using the asymptotic behaviour of the Ψ_k. For example,

$$\dot{\Psi}_2^0 - \sigma^0\Psi_4^0 + \eth\Psi_3^0 = 0. \tag{6}$$

This allows us to define a mass which is monotonic on $\mathscr{I}^+$. First, we define the *mass*

[3] We use the spin weighted derivative $\eth$. On global projective coordinates π^i for $\mathbb{CP}^1$ f has spin weight s if $f(\lambda\pi, \bar{\lambda}\bar{\pi}) = \lambda^{-s}\bar{\lambda}^s f(\pi, \bar{\pi})$. Multiplication by $(\pi^i\bar{\pi}^i)^{-s}$ is an isomorphism with the homogenous holomorphic functions of weight $-2s$. Then $\eth$ can be defined as ∂ acting on the homogenous weight $-2s$ functions corresponding to the spin weight s functions [13]. See chapter 4 of [11] for coordinate expressions. It is the covariant derivative D_z employed by Strominger [4] up to a factor of $2P$.

aspect $\Psi = \Psi_2^0 + \sigma^0 \dot{\bar{\sigma}}^0$ (see also ref. [14]). Using equation (6),

$$\dot{\Psi} = \dot{\sigma}^0 \dot{\bar{\sigma}}^0 - \eth^2 \dot{\bar{\sigma}}^0. \tag{7}$$

Integrating the mass aspect over a $u =$ const. cut we define (in agreement with equation (9.10.9) of ref. [12])

$$M_B(u) \equiv -\frac{1}{4\pi G} \int \mathrm{d}\mu \, \Psi(u, z, \bar{z}),$$

where $\mathrm{d}\mu$ is the measure on the sphere. M_B corresponds to the ADM mass near i^0. (The other $l \leq 1$ components of Ψ give the ADM momentum.) Combining this with equation (7) we find

$$\dot{M}_B \equiv -\frac{1}{4\pi G} \int \mathrm{d}\mu \, \dot{\Psi} = -\frac{1}{4\pi G} \int \mathrm{d}\mu \, \dot{\sigma}^0 \dot{\bar{\sigma}}^0 \leq 0, \tag{8}$$

since $\eth^2 \dot{\bar{\sigma}}^0$ has no $l = 0$ component. This is the mass-loss formula. Notice that $\dot{\sigma}^0 \dot{\bar{\sigma}}^0 / 4\pi G$ can be interpreted as the flux of gravitational radiation at $\mathscr{I}^+$. In the rest of this section we describe the translation to a system of Bondi coordinates. We can use u and r, together with some x^i, to form a system of coordinates. It follows from our discussion above that n has coordinate components

$$n = \frac{\partial}{\partial u} + U \frac{\partial}{\partial r} + X^i \frac{\partial}{\partial x^i},$$

for some functions U, X^i. The $O(r^0)$ component of U is, after applying the field equations, equal to ρ_0'. (In particular, see equation (10l) and the ensuing discussion in ref. [10]). By orthogonality with l, we find that m takes the form

$$m = \omega \frac{\partial}{\partial r} + \zeta^i \frac{\partial}{\partial x^i}.$$

In these coordinates, we can write the inverse metric $g^{ab} = 2(l^{(a} n^{b)} - m^{(a} \bar{m}^{b)})$ in terms of the functions U, X^i, ω, ζ^i: which are partially fixed by the field equations. Notice that $g^{00} = g^{0i} = 0$ and $g^{01} = 1$. This makes it easy to compute the metric g_{ab} which must satisfy $g_{11} = g_{1i} = 0$ and $g_{01} = 1$. To be explicit, $g_{j0} = -g_{jk} g^{k1}$ and $g_{00} = -g^{11} + g^{1i} g_{ij} g^{j1}$, where g_{ij} is the inverse of g^{ij}. Using this and the asymptotic expansions for U, X^i, ω, ζ^i [9] one finds

$$g = \left(-1 - \frac{\mathfrak{Re}\Psi}{r}\right) \mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + 2r^2\gamma_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z} \\ + \left(r\gamma_{z\bar{z}}\bar{\sigma}^0\mathrm{d}z^2 + 2P^{-1}\bar{\eth}\bar{\sigma}^0\mathrm{d}u\mathrm{d}z + \text{c.c.}\right) + \text{subleading}. \quad (9)$$

(We have changed signatures to agree with ref. [4].) We will make particular use of

$$\bar{\sigma}^0 = \lim_{r\to\infty} \frac{1}{r}\frac{P^2}{2} g_{zz}, \quad (10)$$

to infer the asymptotic shear from the metric in Bondi gauge.

2.3 Geodesic Deviation

Timelike geodesics near $\mathscr{I}^+$ are to a good approximation given by lines of constant $(r, z, \bar{z})$ in some Bondi frame. From equation (9) one sees that changes in the separations of these worldlines will arise as σ^0 evolves with u. In this section, we will describe the geodesic deviation of timelike geodesics near $\mathscr{I}^+$ and confirm that the leading order deviation is shearing proportional to σ^0. Consider a geodetic timelike congruence with tangent vector V^a normalised so that $V^aV_a = 1$. We proceed in two steps: (i) choosing a conformal factor adapted to the congruence, and (ii) choosing a null tetrad. Let $\hat{V}^a$ be the smooth extension of V^a to the compactified spacetime and notice that $\hat{V}_a\hat{V}^a = \Omega^2$. It follows[4] that $\hat{V}^a \approx \mathcal{A}\hat{N}^a$ for some non-vanishing scalar $\mathcal{A}$. We rescale Ω so that $\hat{V}^a \approx \hat{N}^a$ on $\mathscr{I}^+$ and we choose to extend Ω off $\mathscr{I}^+$ such that

$$V^a = N^a + O(\Omega^2) \quad \text{and} \quad V^a\hat{N}_a = 0. \quad (11)$$

[4]We have $\hat{V}^a = V^a$ on int$\hat{M}$ and $V^2 = 1$ on int$\hat{M}$ so that $\hat{V}^a$ is non-vanishing on $\partial\hat{M}$. Then the pullback of $\hat{V}$ to $T\mathscr{I}^+$ by the inclusion is non-vanishing and null. But the induced metric on $\mathscr{I}^+$, given in §3, has only one degenerate direction: $\partial/\partial u$.

That is, Ω is constant along the timelike geodesics[5]. Given this choice

$$\hat{N}^a \hat{N}_a = -\Omega^2 + O(\Omega^3).$$

which follows from $(\hat{V}^a - \hat{N}^a)(\hat{V}_a - \hat{N}_a) = O(\Omega^4)$. We also find

$$V^a \nabla_a N^b = O(\Omega^2), \tag{12}$$

which follows by writing $V^a = N^a + \Omega^2 Q^a$ and acting with the covariant derivative. For our purposes, we choose a tetrad (in Bondi gauge) scaled so that $\hat{l}^a \hat{\nabla}_a \Omega \approx -1$. By standard arguments, one finds $\hat{N}_a = (1 + O(\Omega^2))\hat{n}_a + O(\Omega^3)$ (this is derived in appendix A). It then follows from (11)(a) that $V^a = n^a + O(\Omega^2)$. Condition (11)(b) further implies that, in Bondi coordinates, n^a does not have a $\partial/\partial r$ component at leading order. In the notation of §2.2 $U = O(\Omega)$ and so $\rho'^0 = 0$. We now use this frame to study the geodesic deviation of the timelike congruence[6]. Let s^a be a connecting vector orthogonal to V^a and satisfying $\pounds_V s^a = 0$. We write its tetrad components as $s^a = al^a + bn^a + \bar{z}m^a + z\bar{m}^a$. The orthogonality condition implies that $a = O(\Omega^2)$ and, using (11)(b), $V^b \nabla_b a = O(\Omega^2)$. All that remains is to expand $\pounds_V s^a = 0$, which can be written as

$$V^b \nabla_b s^a = s^b \nabla_b V^a.$$

We evaluate the left hand side to find

$$\text{LHS} = V^b \nabla_b s^a = \dot{b} n^a + b D' n^a + (\dot{\bar{z}} m^a + \bar{z} D' m^a + \text{c.c.}) + O(\Omega^2).$$

For the right hand side, notice that $s^a \nabla_a V^b = s^a \nabla_a n^b + O(\Omega^2)$. So

$$\text{RHS} = \bar{z} \delta n^a + z \delta' n^a + O(\Omega^2).$$

[5]Ludvigsen [15] chooses Ω to satisfy $V^a \hat{N}_a = 0$ and $V^a \nabla_a N^b = 0$. However, his argument still holds for $V^a \nabla_a N^b = O(\Omega^2)$. Our conditions, though weaker, do imply $V^a \nabla_a N^b = O(\Omega^2)$ and so permit the same analysis.

[6]We are doing little more than deriving the propagation equations for a null congruence in a special case. The general solution is not much more complicated: see §7.1 of Penrose-Rindler [12].

Here we have used $D'n^a = O(\Omega^2)$, which follows from equation (12). Contracting both sides with l^a gives

$$\dot{b} = \bar{z}(l^a \delta n_a - l^a D' m^a) + z(l^a \delta' n_a - l^a D' \bar{m}^a).$$

In spin-coefficients, $l^a \delta n_a - l^a D' m^a = \alpha' + \bar{\beta}'$. So, referring to equations (3)-(5) we find $\dot{b} = O(\Omega^2)$. Similarly, contracting with $\bar{m}^a$ gives

$$-\dot{z} = \bar{z}(-m_a D' m^a + m_a \delta n^a) + z(-m_a D' \bar{m}^a + m_a \delta' n^a),$$

where we can identify the expansion scalar $\rho' = m^a \delta' n_a$ and the shear $\sigma' = m^a \delta n_a$. Clearly $m_a D' m^a$ vanishes. Moreover, in spin coefficients, $m_a D' \bar{m}^a$ is $\gamma + \bar{\epsilon}'$, which vanishes to order $O(\Omega^2)$. So referring to equations (3)-(5) we find that

$$\dot{z} = \Omega \dot{\bar{\sigma}}^0 \bar{z} + O(\Omega^2). \tag{13}$$

Having thus found the evolution of the connecting vector s^a, we can determine the changes in the proper separations of the geodesics. To first order in Ω we can integrate equation (13) between some u_i and u_f to find a change

$$\Delta z = \Omega \Delta \sigma^0 \bar{z} + O(\Omega^2).$$

The norm of s^a at u_i is

$$s_i^2 = -2(l_{(a} n_{b)} - m_{(a} \bar{m}_{b)}) s_i^a s_i^b = 2 z \bar{z} + O(\Omega^2),$$

for some initial value of z. So the change in proper distance is

$$\Delta s = \frac{s_f^2 - s_i^2}{s_f + s_i} = \frac{\Omega}{s_i}(\Delta \sigma^0 \bar{z}^2 + \Delta \bar{\sigma}^0 z^2) + O(\Omega^2),$$

which we can write as

$$\frac{\Delta s}{s_i} = \frac{\Omega}{2}\left(\Delta\sigma^0\frac{\bar{z}}{z} + \Delta\bar{\sigma}^0\frac{z}{\bar{z}}\right). \tag{14}$$

This agrees with equation (4.5) in ref. [3].

3 Asymptotic Symmetries

The scattering of massless particles is naturally formulated in terms of data supported at $\mathscr{I}^\pm$. We thus anticipate that we should study symmetries of null infinity in order to derive consequences for the scattering problem. Recall that $\hat{g}_{ab}$, and thus the metric it induces on $\mathscr{I}$, is only defined up to a conformal rescaling of Ω. That is, the symmetries of $\mathscr{I}$ are not the isometries, but rather the conformal isometries of the metric on $\mathscr{I}$. We call V an *asymptotic conformal Killing vector* (CKV) if $\pounds_V g_{ab} = \phi g_{ab} + O(\Omega)$ for smooth ϕ. If $\hat{V}$ is an extension of this V to $\hat{M}$, then $\pounds_{\hat{V}}\hat{g}_{ab} \approx V(\phi)\hat{g}_{ab}$. Thus, the pullback of $\hat{V}$ to $\mathscr{I}^+$ by the inclusion $\mathscr{I}^+ \hookrightarrow \hat{M}$ is a CKV of $\mathscr{I}^+$. In this way, asymptotic CKVs induce symmetry transformations on $\mathscr{I}$. However, given a CKV on $\mathscr{I}$ there may be many ways to extend it to an asymptotic CKV on the original manifold (M, g). Let us now specialise to Bondi gauge. We would like to find all the conformal transformations of $\mathscr{I}^+$ that preserve the gauge conditions. The induced metric on $\mathscr{I}^+$ is the pullback of $\hat{g}_{ab}$ by the inclusion map. We use equation (9) to find that induced metric is

$$h = 2\gamma_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}.$$

Notice that it is degenerate since $h(\partial_u, X) = 0$ for all vectors X. We see that h is invariant (up to conformal scaling) under (i) the 2d conformal transformations of the sphere, and (ii) any replacement of the form $u \mapsto f(u, z, \bar{z})$. Consider case (i). These are the Mobius transformations $z \mapsto z' = (az+b)/(cz+d)$ (for any $a, b, c, d \in \mathbb{C}$ satisfying $ad - bc = +1$). Under this replacement, h transforms as $h \mapsto \omega^2 h$ where

$$\omega = \frac{1 + z\bar{z}}{|az+b|^2 + |cz+d|^2}.$$

By construction, this is accompanied by a rescaling of the conformal factor: $\Omega \mapsto \omega\Omega$. Thus, in order to preserve the gauge condition $D'u \approx 1$, we must demand that the Bondi parameter transforms as $u \mapsto \omega u$. We now consider case (ii). Recall the condition that Bondi parameters must satisfy $D'^2 u \approx 0$. This restricts us to transformations of the form $u \mapsto g(z,\bar{z})u + h(z,\bar{z})$. However, we are further restricted by the condition $D' \approx 1$. So, unless we simultaneously perform a transformation of type (i), we must have $g(z,\bar{z}) = 1$. Thus, the gauge preserving conformal transformations are

$$z' = \frac{az+b}{cz+d}, \quad u' = \frac{1+z\bar{z}}{|az+b|^2 + |cz+d|^2} u + h(z,\bar{z}). \tag{15}$$

This is called the BMS group, $\mathcal{B}$. We can write it as a product $\mathcal{B} = \mathcal{S} \ltimes SL(2)_{\mathbb{C}}$ where $\mathcal{S}$ is the normal subgroup of *supertranslations*: $u \mapsto u + h(z,\bar{z})$. Much like the Poincaré group, while $\mathcal{B}/\mathcal{S} \simeq SL(2)_{\mathbb{C}}$, we cannot identify a canonical Lorentz subgroup. Indeed, any given $\mathcal{L} \subset \mathcal{B}$ isomorphic to $SL(2)_{\mathbb{C}}$ can be transformed to a distinct subgroup $s^{-1}\mathcal{L}s \simeq SL(2)_{\mathbb{C}}$, for some nontrivial $s \in \mathcal{S}$. In the Poincaré group, each Lorentz subgroup corresponds to a choice of origin in Minkowski space. For $\mathcal{B}$ we see that each subgroup corresponds to a cut of $\mathscr{I}^+$. That is, the Lorentz subgroups can be parameterised by the functions $f(z,\bar{z})$ on the sphere. On the other hand, we can identify a subgroup of $\mathcal{T} \subset \mathcal{S}$ that we call *translations*. This is motivated by considering Minkowski space. Here, the Bondi parameters u (up to scaling) correspond to a choice of constant timelike vector v (and hence a choice of non-intersecting light cones that foliate $\mathscr{I}^+$). Let p^a be a future-pointing null vector corresponding to the point z on the sphere (unique up to scaling). We define the u coordinate of the point on $\mathscr{I}^+$ reached by the null ray through x with tangent p as $u = p \cdot x / p \cdot v$. So, take coordinates x^a such that $v = \partial/\partial x^0$. Then $p^a \propto (1 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z})$ and under a translation, $x^a \mapsto x^a + a^a$, the retarded time transforms as

$$u \mapsto u + K, \quad \text{where} \quad K = a^0 + (a^1 - ia^2)\frac{z}{1+z\bar{z}} + (a^1 + ia^2)\frac{\bar{z}}{1+z\bar{z}} - a^3\frac{(z\bar{z}-1)}{1+z\bar{z}}.$$

That is, translations correspond to supertranslations of u by the $l \leq 1$ spherical harmonics. For this reason we define the translation subgroup $\mathcal{T} \subset \mathcal{S}$ of a general asymptotically flat spacetime as supertranslations by the $l \leq 1$ spherical harmonics. This means that energy and momentum conservation can be defined. On account of the large number of Lorentz subgroups, angular momentum conservation is more subtle[7] [17].

A Note on Coordinate Transformations

While the transformations described above are conformal isometries of $\mathscr{I}^{\pm}$, they do not specify how the Bondi coordinates on the neighbourhood of $\mathscr{I}$ should transform in order for our coordinate expressions to remain in Bondi gauge. Alternatively, we could ask how the tetrad associated to a system of Bondi coordinates ought to transform in the neighbourhood of $\mathscr{I}$. Let us focus on supertranslations. Consider a vector field V whose pullback to $\mathscr{I}^{+}$ is a supertranslation generator: $\hat{V}^{*} = f(z, \bar{z})\partial_u$. Then, the requirement that $\pounds_V\, g$ leaves g, equation (9), in the form of a Bondi metric uniquely determines V up to the addition of subleading terms. One finds

$$V = f\partial_u - \frac{1}{r}\gamma_{z\bar{z}}^{-1}\bar{\partial}f\partial - \frac{1}{r}\gamma_{z\bar{z}}^{-1}\partial f\bar{\partial} + \gamma_{z\bar{z}}^{-1}\partial\bar{\partial}f\partial_r.$$

The associated coordinate transformations are $\delta z = -r^{-1}\gamma_{z\bar{z}}^{-1}\bar{\partial}f$ and $\delta r = \gamma_{z\bar{z}}^{-1}\partial\bar{\partial}f$. Moreover, the mass aspect transforms as $\delta\Psi = f\dot{\Psi}$ and the asymptotic shear transforms as $\delta\sigma^0 = -2\gamma_{z\bar{z}}^{-1}\partial^2 f + f\dot{\sigma}^0$. For a finite supertranslation, we see that the Bondi frame associated to the advanced time $u' = u + f$ has asymptotic shear

$$\sigma'^0(u) = \sigma^0(u) - 2\gamma_{z\bar{z}}^{-1}\partial^2 f.$$

For simplicity, we have given σ'^0 at the point $(u, z, \bar{z})$ in the original coordinates. The function $\sigma'^0(u', z', \bar{z}')$ can be found by integrating the infinitesimal transformations above.

[7] Penrose and Newman [16] proposed the condition $\sigma^0 \to 0$ as $u \to -\infty$. This fixes the supertranslation freedom (leaving translations) and thus defines a Poincaré group. They argued that this is in fact a coordinate choice in this case of linearised gravity (i.e. not a restriction on the allowed spacetimes). One could then fix an $SU(2)$ subgroup by choosing some canonical timelike vector like the ADM momentum.

In terms of the tetrad associated to the coordinates one finds that a supertranslation induces a null rotation of the tetrad around n^a given by $\iota^A \mapsto \iota^A$ and $o^A \mapsto o^A - r^{-1}\eth f$. One can then compute the same transformation for σ^0 (see, e.g., §1.3 of [18]).

4 The Graviton Soft Factor

Consider the scattering of n gravitons. In the soft limit – in which the energy of the n^{th} graviton is small – the n-particle amplitude factorises into the product of the corresponding $(n-1)$-particle amplitude and a *soft factor* [19]. In this section we derive this result. Let $\kappa = \sqrt{32\pi G}$ be the gravitational coupling constant. The Einstein-Hilbert action is

$$S_{EH} = \int \mathrm{d}^4 x \, \frac{2}{\kappa^2} \sqrt{-g} R. \tag{16}$$

To make our calculation as efficient as possible, we take $\tilde{g}^{ab} = \sqrt{-g} g^{ab}$ as our dynamical field[8]. This gives a simple expression for the Lagrangian, since

$$\sqrt{-g}R = \frac{1}{4}\left(\tilde{g}^{ab}\tilde{g}_{cm}\tilde{g}_{dn} - \frac{1}{2}\tilde{g}^{ab}\tilde{g}^{cd}\tilde{g}^{mn} - 2\delta^b_d \delta^a_m \tilde{g}_{cn}\right)\partial_a \tilde{g}^{cd}\partial_b \tilde{g}^{mn} \tag{17}$$

We perform an expansion $\tilde{g}^{ab} = \eta^{ab} + \kappa\phi^{ab}$ (such that ϕ^{ab} is dimensionful and has kinetic terms that do not involve the coupling). It is easy to identify the cubic terms in equation (17) and read off the vertex factor. For three incoming modes with momenta p^i and indices $a_i b_i$ we have

$$\begin{aligned} V_{a_1b_1a_2b_2a_3b_3} = -\frac{i\kappa}{2}\Bigg[& p^1_{a_3}p^2_{b_3}\left(2\eta_{a_1a_2}\eta_{b_1b_2} - \frac{1}{2}\eta_{a_1b_1}\eta_{a_2b_2}\right) + 2q^2_{a_1}q^1_{b_2}\eta_{b_1a_3}\eta_{b_3a_2} \\ &+ q^1\cdot q^2\left(\frac{1}{2}\eta_{a_1b_1}\eta_{a_2a_3}\eta_{b_2b_3} + \frac{1}{2}\eta_{a_2b_2}\eta_{a_1a_3}\eta_{b_1b_3} - 2\eta_{b_1a_2}\eta_{a_1a_3}\eta_{b_2b_3}\right)\Bigg] + \text{even perms.} \end{aligned} \tag{18}$$

In this expression, we implicitly symmetrise over all index pairs $a_i b_i$. To define a propagator, a partial gauge fixing must be performed (as in quantum electrodynamics). We

[8] This introduces a functional determinant in the path integral which is absorbed into the normalisation.

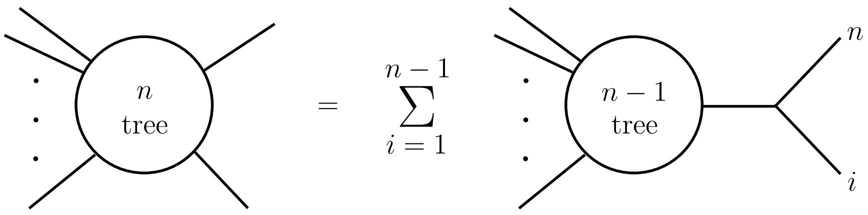

Figure 1: The leading tree-level contributions to the tree-level amplitude in the soft limit.

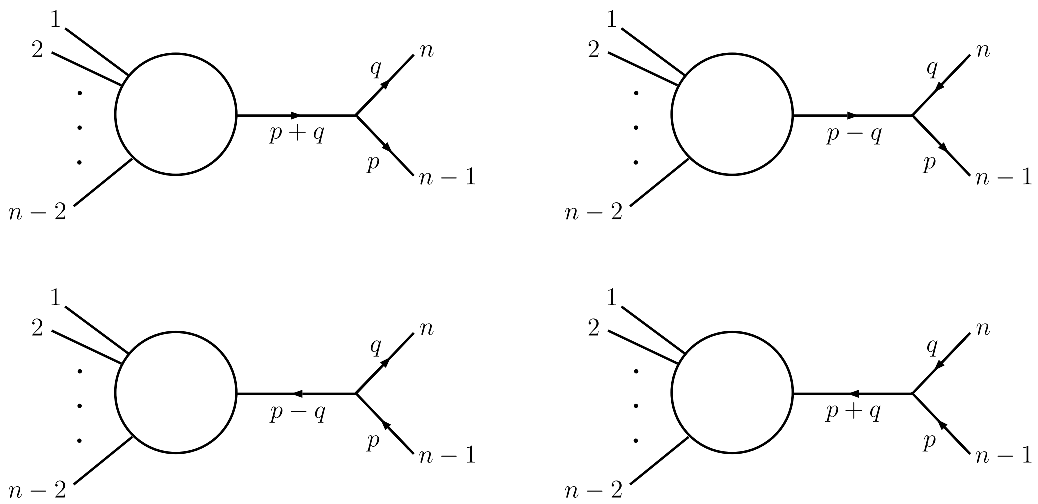

Figure 2: One tree-level contribution shown for the four cases in which the soft graviton is incoming/outgoing and the hard graviton is incoming/outgoing.

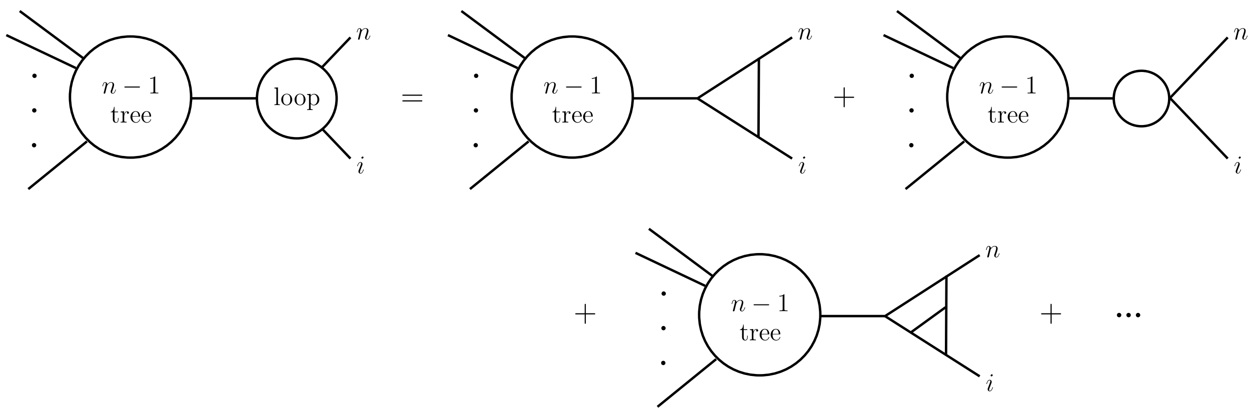

Figure 3: The leading factorizable loop corrections to the soft vertex.

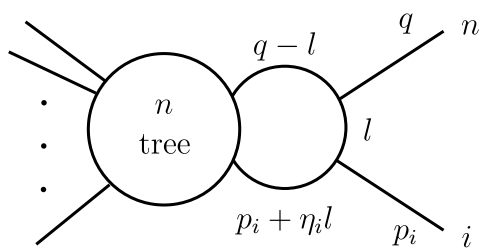

Figure 4: The leading order non-factorizable loop correction.

leave details of this to ref. [20] and choose the gauge in which the propagator takes the form:

$$D_{aba'b'}(k) = \frac{-i}{2k^2}\left(\eta_{aa'}\eta_{bb'} + \eta_{ab'}\eta_{ba'} - \eta_{ab}\eta_{a'b'}\right).$$

Let $\mathcal{M}_n$ be the scattering amplitude for n gravitons with momenta p^i and polarisations ϵ^i. For the n^{th} graviton we will write $p^n = q$ and $\epsilon^n = \epsilon$ and consider $\mathcal{M}_n$ in the soft limit $q \to 0$. To be systematic, consider first the tree-level amplitude $\mathcal{M}_{n,\text{tree}}$ in the soft limit. The leading contributions come from diagrams where the soft graviton is attached to external on-shell momenta by a cubic vertex: see figure 1. These diagrams have a pole since they include a propagator with momentum $p^i \pm q$ for some on-shell p^i. That is, they include a factor of $1/(p^i \pm q)^2 = \pm 1/(2q \cdot p^i)$. These are the only tree-level diagrams with soft poles. In all other diagrams the soft graviton is either attached to an internal line via a cubic vertex or attached to any line via a vertex that is quartic or higher. In any of these diagrams, none of the propagators attached to the soft vertex have momentum $p \pm q$ for on-shell p: so the $q \cdot p$ pole does not arise. We can thus compute the leading order contribution to $\mathcal{M}_{n,\text{tree}}$ in the soft limit:

$$\begin{aligned}\mathcal{M}_{n,\text{tree}} &= \sum_i \mathcal{M}^{cd}_{n-1,\text{tree}}(p^i + \eta^i q) D_{cdc'd'}(p^i + \eta^i q) V^{c'd'mnab} \epsilon^a_{mn} \epsilon_{ab} + \text{subleading} \\ &= -i \sum_i \mathcal{M}^{cd}_{n-1,\text{tree}}(p^i + \eta^i q) \left(2V_{cd}{}^{mnab} - \eta_{cd} V_e{}^{emnab}\right) \frac{\epsilon^i_{mn}\epsilon_{ab}}{4\eta^i p^i \cdot q} + \text{subleading}.\end{aligned}$$

We write $\mathcal{M}^{cd}_{n-1,\text{tree}}(p^i + \eta^i q)$ for the $(n-1)$-graviton amplitude with momenta p^j for $j \neq i$ and p^i replaced by $p^i + \eta^i q$. Here, cd is the index pair for the i^{th} particle. We define η^i to be a sign that is positive if the soft graviton and q^i are both outgoing or both ingoing; else it is negative. (See figure 2 for the diagrams corresponding to these cases.) In the soft limit, the vertex factor with momenta $(p^i + \eta^i q, p^i, q)$ simplifies to

$$V^{cdmnab} = \frac{i\kappa}{2}\left[p_i^a p_i^b \left(2\eta^{cm}\eta^{dn} - \frac{1}{2}\eta^{cd}\eta^{mn}\right) + 2p_i^c p_i^n \eta^{da}\eta^{bm}\right] + O(q).$$

Notice that equation (18) was quoted for incoming momenta: so we have picked up a minus sign to account for the fact that either $p^i + \eta^i q$ or p^i is outgoing. Using $\epsilon^i_{ab}\eta^{ab} = \epsilon^i_{ab} p_i^b = 0$, we find

$$\mathcal{M}_{n,\text{tree}} = \frac{\kappa}{2}\sum_i \frac{\eta^i p_a^i p_b^i \epsilon^{ab}}{p\cdot q}\mathcal{M}_{n-1,\text{tree}}(p^i+\eta^i q) + O(q^0).$$

On account of the simple structure of tree amplitudes we can write $\mathcal{M}_{n-1,\text{tree}}(p^i+\eta^i q) = \mathcal{M}_{n-1,\text{tree}} + O(q)$. We then find

$$\mathcal{M}_{n,\text{tree}} = \frac{\kappa}{2}\sum_i \frac{\eta^i p_a^i p_b^i \epsilon^{ab}}{p\cdot q}\mathcal{M}_{n-1,\text{tree}} + O(q^0).$$

Surprisingly, this result is tree-level exact. To see this, consider just one of the diagrams that appears in the tree-level sum (figure 1). We can consider three types of loop corrections to this term: (i) factorizable corrections to $\mathcal{M}_{n-1,\text{tree}}$, (ii) factorizable corrections to the soft vertex, and (iii) non-factorizable corrections. Here, a 'factorizable' diagram is one that has a propagator which, if removed, would give two disconnected diagrams, with one diagram involving only p^a and the soft graviton. We claim that diagrams of type (ii) and (iii) do not contribute at leading order in the soft limit. This is easy to see for case (ii). All diagrams of this type have a coupling factor of κ^k for $k \geq 3$. (The lowest order terms are shown in figure 3.) On dimensional grounds, since $[\kappa] = -1$, these diagrams must have a factor of $(q\cdot p^a)$ or higher compared to the tree-level diagram: and so they are subleading in the soft limit. Case (iii) is more involved. The leading order 1-loop non-factorizable contribution is given in figure 4. The contribution of this diagram is

$$\int \widetilde{\mathrm{d}}l\mathcal{M}_{n,\text{tree}}(q-l, p^i+\eta^i l)D(q-l)V(-q,q-l,l)D(l)V(p^i,\eta^i l,-p^i-\eta^i l)D(p^i+\eta^i l)\epsilon\epsilon^i,$$

where we are suppressing the Lorentz indices and writing the loop integration measure as $\widetilde{\mathrm{d}}l$. (Notice that the momenta have been chosen to match our convention that V is for all-incoming momenta.) By power counting, some of the summands in this integrand have an infrared divergence. The finite integrals do not have a soft pole, so we focus on checking the infrared divergent term. We expand the vertex functions in the loop momentum l to find, after applying $q_{c'}\epsilon^{cc'} = \eta_{cc'}\epsilon^{cc'} = 0$, that

$$V(-q,q-l,l)\epsilon = i\kappa q_m q_{m'}\eta_{ac}\eta_{a'c'}\epsilon^{cc'} + O(l),$$

where the mm' indices connect to the l propagator and the aa' indices connect to the $q-l$ propagator. The result for the other vertex is similar. We expand the tree-level amplitude, $\mathcal{M}_{n,\text{tree}}(q+l, p^i+\eta^i l) = \mathcal{M}_{n,\text{tree}} + O(l)$, to find that the infrared divergent part is

$$-i\kappa^2(q\cdot p^a)^2\mathcal{M}_{n,\text{tree}}\int \widetilde{\mathrm{d}}l\frac{1}{(l-q)^2l^2(l+\eta^a p^a)^2}.$$

The integral is standard: in $d=4-2\epsilon$ dimensions it has a ϵ^{-2} divergence and it is proportional to $(q\cdot p^a)^{-(1+\epsilon)}$ (see [21]). So, the term is proportional to $q\cdot p^a$ and, in the soft limit, it contributes only at sub-subleading order. Bern [22] argues that higher order and n-loop non-factorizable contributions are also subleading due to the powers of q and l that appear in the vertices[9]. We conclude that only loop corrections of type (i) contribute at leading order in the soft limit; it follows that

$$\mathcal{M}_n = \frac{\kappa}{2}\sum_a \frac{\eta^a p^a_\alpha p^a_{\alpha'}\epsilon^{\alpha\alpha'}}{p\cdot q}\mathcal{M}_{n-1} + O(q^0). \tag{19}$$

This is the graviton soft factorisation theorem. Precisely the same factor is found for the emission of soft gravitons by scattering scalars or fermions. These particles can be massive. In this case attaching the soft graviton to external lines gives factors of the form $[(\eta q+p)^2+m^2]^{-1} = [2\eta\, q\cdot p]^{-1}$, since the external momenta are on-shell. Thus the same pole arises that we found for graviton amplitudes. A similar factor can be found for soft photons and soft gluons. However, for gluons the non-factorisable infrared divergences discussed above contribute at leading order. It was recently found [5] that the graviton soft factorisation, equation (19), is equivalent to the Ward identity for the group of diagonal supertranslations. This is the subgroup of $\mathcal{S}^+\times\mathcal{S}^-$ comprising supertranslations that are the same on $\mathscr{I}^+$ and $\mathscr{I}^-$: given that null generators are identified at i^0. These diagonal supertranslations can be generated by operators $Q(f)$ on the state space of asymptotic states. This state space was originally formulated by Ashtekar [24, 25] and admits a symplectic structure. Since the operators $Q(f)$ commute amongst themselves, and since the Hamiltonian can be identified as a generator $Q(g)$ for some timelike $g\in\mathcal{T}$, Strominger

[9] In fact, contributions at $O(q^0)$ are from at most 1-loop diagrams, and $O(q)$ from at most 2-loop diagrams. [23].

conjectured that $[Q(f), \hat{S}] = 0$, where $\hat{S}$ is the (unknown) gravity S-matrix. The matrix elements of $[Q(f), \hat{S}]$ then yield a Ward identity that is equivalent to the soft factorisation (after performing two derivatives).

5 The Memory of Graviton Scattering

The displacement of neighbouring masses near $\mathscr{I}^{\pm}$ is related to changes in the asymptotic shear σ^0 by equation (14). It is the purpose of this section to derive this change in σ^0 for graviton scattering. Earlier we found a field equation, equation (7), which can be integrated to give [14]

$$\eth^2 \Delta^+ \bar{\sigma}^0 = \int \mathrm{d}u \, \dot{\sigma}^0 \dot{\bar{\sigma}}^0 - \Delta^+ \Psi. \tag{20}$$

We write $\Delta^+ \bar{\sigma}^0 = \int \mathrm{d}u \, \dot{\bar{\sigma}}^0$ for the change on $\mathscr{I}^+$. This equation is equivalent to Christodoulou's system of equations (10)-(12) in ref. [26]. There is a similar equation for $\mathscr{I}^-$. The right hand side has mass dimension +1 and we might guess that for graviton scattering it takes the form $\sum E_i \, \delta^2(z - z_i)$, for some energies E_i. This guess is essentially correct. At first we will derive this using linearised gravity in the spirit of the original literature [1,27,28]. Of course, linearised gravity is not valid for hard gravitons: so we include hard gravitons as a contribution to the stress tensor. This naïve calculation is performed on globally flat space. Surprisingly, it is possible to rederive the same result under the much weaker assumption of asymptotic flatness. We do this in §5.2.

5.1 Memory from Linearised Gravity

Einstein's equation follows from equation (16) and is $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 0$. We expand $g_{ab} = \eta_{ab} + h_{ab}$ and write $G_{ab} = G^{(0)}_{ab} + G^{(1)}_{ab} = 0$, where $G^{(0)}_{ab}$ is the linear contribution. Using diffeomorphism invariance we choose a gauge in which h_{ab} is traceless ($\eta^{ab} h_{ab} = 0$) and transverse ($\partial^a h_{ab} = 0$). Then $G^{(0)}_{ab} = -\frac{1}{2} \partial^2 h_{ab}$, where $\partial^2 = \eta^{mn} \partial_m \partial_n$. Recall that the advanced and retarded Green's functions for $-\partial^2$ are

$$K^{\pm}(x,y) = \frac{1}{4\pi\,|\mathbf{x}-\mathbf{y}|}\,\delta(x^0 - y^0 \pm |\mathbf{x}-\mathbf{y}|),$$

with $-\partial^2 K^{\pm}(x,y) = \delta(x-y)$. The delta function has support on the past (future) light cone of x. Near $\mathscr{I}$, gravitons are momentum eigenstates and we can describe them using an ansatz for the effective stress tensor: $\kappa^2 T_{ab}/2 = G^{(1)}_{ab}$. (We are being consistent with the normalisation implied by equation (16).) In transverse-traceless gauge there is only one term at second order:

$$G^{(1)}_{ab} = \frac{1}{4} h_{kl,a} h_{k'l',b} \eta^{kk'} \eta^{ll'} + O(h^3).$$

For momentum eigenstates near $\mathscr{I}^{\pm}$, $h_{ab} \sim \Omega\,\epsilon_{ab}\,e^{iEk\cdot x}$, where k^a is a future-pointing null vector normalised by $k^0 = 1$. By our gauge choice we require $\epsilon_{ab}k^b = \epsilon_{ab}\eta^{ab} = 0$. Substituting this above we infer that the effective stress tensor has the form $T_{ab} = k_a k_b T_{00}$, where $T_{00} = O(\Omega^2)$[10]. We could write $T_{00} = \Omega^2 \mathcal{F} + O(\Omega^3)$ where $\mathcal{F}$ is a function on $\mathscr{I}^+$. But T_{ab} is a stress tensor, so we expect the integral of $\mathcal{F}$ over $\mathscr{I}^+$ to give the total energy[11]. Thus, we take our ansatz for a single outgoing graviton incident on $\mathscr{I}^+$ to be

$$T_{00}(u,\hat{\mathbf{x}}) = -\Omega^2 E \delta(u-u_*)\delta^{(2)}(\hat{\mathbf{x}} - \mathbf{k}),$$

for some u_*. We want to calculate the change in asymptotic shear, $\Delta\sigma^0 = \int \mathrm{d}u \dot{\sigma}^0$. To infer $\Delta\sigma^0$ on $\mathscr{I}^+$ we study the leading contributions to h_{ab} for large $r = |\mathbf{x}|$ where this limit is taken with fixed $u = x^0 - r$. We begin by computing the contribution to h_{ab} from an outgoing graviton, which comes from T_{ab} near $\mathscr{I}^+$. Recalling the retarded Green's function K^- we have

$$h_{ab}(x) = 4G \int\limits_{\mathcal{I}^-(x)} \mathrm{d}^4x'\, T_{ab}(x') \frac{1}{|\mathbf{x}-\mathbf{x}'|}\,\delta(x^0 - x'^0 - |\mathbf{x}-\mathbf{x}'|).$$

[10] To see how this can also be inferred from the dominant energy condition see ref. [29].
[11] In Christodoulou's notation [2], one has $\int \mathrm{d}u\mathcal{F} = F/4\pi$. In ref. [27] one has $\mathcal{F} = \mathrm{d}E/\mathrm{d}\Omega$.

We write $u' = x'^0 - r'$ so that

$$h_{ab}(x) = 4G \int_{-\infty}^{u} \mathrm{d}u' \int r'^2 \mathrm{d}r' \int \mathrm{d}\mu' \, T_{ab}(x') \frac{1}{|\mathbf{x} - \mathbf{x}'|} \, \delta(r + u - r' - u' - |\mathbf{x} - \mathbf{x}'|).$$

The stress tensor fixes the angular integral and the integral over u'. So

$$h_{ab}(x) = -4G\Theta(u - u_*) \int \mathrm{d}r' \frac{Ek_a k_b + O(r^{-1})}{|\mathbf{x} - r'\mathbf{k}|} \delta(r + u - r' - u_* - |\mathbf{x} - r'\mathbf{k}|).$$

A short calculation gives

$$\frac{\partial}{\partial r'} \left(-r' - |\mathbf{x} - r'\mathbf{k}|\right) = \frac{1}{|\mathbf{x} - r'\mathbf{k}|} \left(-r' + \mathbf{x} \cdot \mathbf{k} - |\mathbf{x} - r'\mathbf{k}|\right). \tag{21}$$

Given this, we can fix r' using the delta function to find[12]

$$h_{ab}(u, z, \bar{z}) = -4G\Theta(u - u_*) \frac{Ek_a k_b + O(r^{-1})}{-r - u + u_* + r\hat{\mathbf{x}} \cdot \mathbf{k}}$$

We can determine the contribution from incoming gravitons near $\mathscr{I}^-$ in a similar way. For incoming gravitons we take T_{00} to be positive (rather than negative as before). Our convention is that all momenta are future-pointing, and so, because $x' - x$ is past-pointing for x' near $\mathscr{I}^-$ and x near $\mathscr{I}^+$ we must be careful when comparing the direction $\hat{\mathbf{x}}'$ with the direction $\mathbf{k}$. Indeed, for x near $\mathscr{I}^+$ we must take, for incoming gravitons,

$$T_{00}(u', \hat{\mathbf{x}}') = \Omega^2 E \delta(u' - v_*) \delta^{(2)}(\hat{\mathbf{x}}' + \mathbf{k}_*).$$

Being careful about the sings that appear in equation (21) we find, for a single incoming graviton,

$$h_{ab}(x) = 4G\Theta(v_* - u) \frac{Ek_a k_b + O(r^{-1})}{r + u - v_* - r\hat{\mathbf{x}} \cdot \mathbf{k}} + O(r^{-2}).$$

[12] As pointed out by ref. [29] this result is singular when $\hat{\mathbf{x}} = \mathbf{k}$. So one might prescribe that we perform the integration only for $\hat{\mathbf{x}} \neq \mathbf{k}$. The appearance of a pole is not surprising for a scattering problem and we will recover the same pole in §5.2.

So, for multiple gravitons with energies E^i and momenta $p^i = E^i(1, \mathbf{k}^i)$, the total change in the asymptotic metric near $\mathscr{I}^+$ is

$$\Delta^+ h_{ab}(\hat{\mathbf{x}}) = \int \mathrm{d}u \, \dot{h}_{ab}(u + r, r\hat{\mathbf{x}}) = \frac{4G}{r} \sum_i \frac{\eta^i p^i_a p^i_b}{q \cdot p^i} + O(r^{-2}), \tag{22}$$

where η^i is $+1$ for outgoing gravitons and -1 for incoming and $q = (1, \hat{\mathbf{x}})$. Since we are working in transverse-traceless gauge, we must understand the RHS to be the transverse-traceless part of the given expression (obtained, e.g., by acting on the spatial components with a projection $\delta_{ij} - \hat{\mathbf{x}}_i \hat{\mathbf{x}}_j$). This result then agrees with [27, 28, 30]. We can determine the change in metric near $\mathscr{I}^-$ in the same way except that we use an advanced propagator. Repeating the calculation one finds

$$\Delta^- h_{ab}(\hat{\mathbf{x}}) = \int \mathrm{d}v \, \dot{h}_{ab}(v - r, r\hat{\mathbf{x}}) = \frac{4G}{r} \sum_i \frac{\eta^i p^i_a p^i_b}{\bar{q} \cdot p^i} + O(r^{-2}),$$

where $\bar{q} = (1, -\hat{\mathbf{x}})$. As mentioned above, one problem with this calculation is that it assumes global flat coordinates. However, we are surprised to find that $\Delta^+ h_{ab}(\hat{\mathbf{x}})$ and $\Delta^- h_{ab}(\hat{\mathbf{x}})$ are the same for antipodal directions on the celestial sphere of a point on the interior of the spacetime. We will find something analogous in §5.2.

5.2 Memory from the Soft Factor

Soft gravitons are large wavelength metric fluctuations. This suggests that we could relate changes in the shear over long distances to soft gravitons. In this section, we find that the change in shear is essentially given by the soft factor. We begin by computing the memory near $\mathscr{I}^+$. We take a Bondi system of coordinates $u, r, z, \bar{z}$. We can associate to this a set of coordinates with $x^0 = u + r$, $x^i = r\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ is a unit spatial vector corresponding to z. In these coordinates the metric is asymptotically Minkowski and we can expand $\kappa h_{ab} = g_{ab} - \eta_{ab}$ into modes,

$$h_{ab}(x) = \kappa \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3(2\omega)} \left(\epsilon^m_{ab} a_m(\mathbf{q}) e^{-iq\cdot x} + \epsilon^{*m}_{ab} a^\dagger{}_m(\mathbf{q}) e^{iq\cdot x} \right),$$

where $q^0 = \omega = |\mathbf{q}|$. Here m is summed over the two helicities. If we write $\omega \cos\theta = \hat{\mathbf{x}} \cdot \mathbf{q}$ then

$$h_{ab}(x) = \frac{\kappa}{2(2\pi)^3} \int_0^\infty \mathrm{d}\omega\, \omega \int \mathrm{d}\mu \, \left(\epsilon^m_{ab} a_m(\mathbf{q})(\mathbf{q}) e^{i\omega u + i\omega r(1-\cos\theta)} + \text{h.c.} \right),$$

where $\mathrm{d}\mu$ is the measure on the unit sphere. We now consider the large r limit with fixed u. By the method of stationary phase, we see that the leading contributions to the angular integral are of order $O(1/r)$ and come from integrating near $\theta = 0$ and $\theta = \pi$, where $f(\theta, \phi) = \omega(1 - \cos\theta)$ is stationary. Near these points we can expand f in local coordinates to find $f \approx \omega(a^2 + b^2)/2 + ...$ near $\theta = 0$ and $f \approx \omega(2 - (a^2 + b^2)/2) + ...$ near $\theta = \pi$. It follows that the Hessian of f has determinant ω at both points, while it has sign $+2$ at $\theta = 0$ and -2 at $\theta = \pi$. Then equation (28) from appendix B gives

$$\int \mathrm{d}\Omega\, a_m(\mathbf{q}) e^{i\omega r(1-\cos\theta)} = \frac{2\pi}{i\omega r} a_m(\omega\hat{\mathbf{x}}) - \frac{2\pi}{i\omega r} e^{2i\omega r} a_m(-\omega\hat{\mathbf{x}}) + O\left(r^{-2}\right).$$

However, the integral $\int_0^\infty \mathrm{d}\omega \exp(2i\omega r)$ is of order $O(r^{-1})$ (see appendix B). So the leading contribution to $h_{ab}(x)$ near $\mathscr{I}^+$ is given only by the $\theta = 0$ contribution. The $a^\dagger$ term is similar (but for a sign) and we find

$$h_{ab}(x) = \frac{1}{r} \frac{\kappa}{2(2\pi)^2 i} \int_0^\infty \mathrm{d}\omega \left(\epsilon^m_{ab} a_m(\omega\hat{\mathbf{x}}) e^{i\omega u} - \epsilon^{*m}_{\ ab} a^\dagger{}_m(\omega\hat{\mathbf{x}}) e^{-i\omega u} \right) + O(r^{-2}).$$

We compute $\Delta^+ h_{ab}(\hat{\mathbf{x}})$ by taking a u-derivative and integrating. The u-integral creates delta functions which each give $1/2$ when integrated over the domain $\omega \in [0, \infty)$. Alternatively, one can see this as a $\omega \to 0^+$ limit of

$$\Delta^+ h^\omega_{ab}(\hat{\mathbf{x}}) \equiv \frac{1}{2} \int \mathrm{d}u \, \left(e^{i\omega u} \dot{h}_{ab}(\hat{\mathbf{x}}) + e^{-i\omega u} \dot{h}_{ab}(\hat{\mathbf{x}}) \right).$$

In this way we find

$$\Delta^+ h_{ab}(\hat{\mathbf{x}}) = \lim_{\omega \to 0^+} \frac{\kappa}{8\pi r} \omega \left(\epsilon^m_{ab} a_m(\omega\hat{\mathbf{x}}) + \epsilon^{*m}_{\ ab} a^\dagger{}_m(\omega\hat{\mathbf{x}}) \right).$$

The $\mathscr{I}^-$ calculation is similar. Near $\mathscr{I}^-$ we take a Bondi system of coordinates $v, r, w, \bar{w}$. From this we construct coordinates with $y^0 = v - r$ and $y^i = r\hat{\mathbf{y}}$. We can do a mode expansion in these coordinates. The function that we previously called f is now $-\omega(1 + \cos\theta)$. This has significant consequences: by the same argument as above, the leading contribution to h_{ab} comes from $\theta = \pi$ (not $\theta = 0$). Moreover, near $\theta = \pi$, the Hessian of f has sign -2 (not $+2$ as before). This introduces an overall change in sign. Our final result is

$$\Delta^- h_{ab}(\hat{\mathbf{y}}) = -\lim_{\omega\to 0^+} \frac{\kappa}{8\pi r}\,\omega\left(\epsilon^m_{ab} b_m(-\omega\hat{\mathbf{y}}) + \epsilon^{*m}_{\ \ ab} b^\dagger{}_m(-\omega\hat{\mathbf{y}})\right). \tag{23}$$

Where $b, b^\dagger$ are the modes near $\mathscr{I}^-$. If we promote the modes to operators via canonical quantisation, the a modes annihilate the out states, while the b modes annihilate the in states. Then, using equation (19) we find

$$\left\langle \Delta^+ h_{ab}(\hat{\mathbf{x}})\right\rangle = \frac{\langle \mathrm{out}|\,\Delta^+\hat{h}_{ab}(\hat{\mathbf{x}})\,|\mathrm{in}\rangle}{\langle \mathrm{out}\,|\,\mathrm{in}\rangle} = \frac{\kappa}{2}\frac{\kappa}{8\pi r}\sum_{i,m}\frac{\eta^i E^i k^i_c k^i_d}{1 - \hat{\mathbf{x}}\cdot\mathbf{k}^i}\epsilon^{m\,cd}(\omega\hat{\mathbf{x}})\epsilon^m_{ab}(\omega\hat{\mathbf{x}}) + O(r^{-2}) \tag{24}$$

where i runs over all hard momenta. This result agrees with the flat space result, equation (22), except that now the transverse-traceless property is neatly given by a projection onto the polarisations. Similarly

$$\left\langle \Delta^- h_{ab}(\hat{\mathbf{y}})\right\rangle = \frac{2G}{r}\sum_{i,m}\frac{\eta^i E^i k^i_c k^i_d}{1 + \hat{\mathbf{y}}\cdot\mathbf{k}^i}\epsilon^{*m\,cd}(-\omega\hat{\mathbf{y}})\epsilon^{*m}_{\ \ ab}(-\omega\hat{\mathbf{y}}) + O(r^{-2}).$$

Notice that the minus sign from the second soft factor cancels with the minus sign in equation (23) so that the overall sign is as for the $\mathscr{I}^+$ result. Our calculation involves two distinct sets of Bondi coordinates for $\mathscr{I}^+$ and $\mathscr{I}^-$. We recall Ashtekar's construction (§2.1) of spacelike infinity i^0. This allows the null generators of $\mathscr{I}^+$ to be identified with those of $\mathscr{I}^-$. We can thus use one set of coordinates on the sphere for both $\mathscr{I}^\pm$. Consider the null ray on $\mathscr{I}^-$ corresponding to the null generator labelled by z on $\mathscr{I}^+$. Motivated by the formulae above we choose to label this by $w = -1/\bar{z}$, such that $\hat{\mathbf{y}} = -\hat{\mathbf{x}}$. In these coordinates it is manifest that the memory along one null generator near $\mathscr{I}^+$ is the same as that for the corresponding generator near $\mathscr{I}^-$.

5.3 Properties of the Memory

A remarkable consequence of the above results is that the displacements of a sphere of detectors in the future of a graviton scattering event are sufficient to determine all scattering data: the energies and directions of both outgoing and incoming particles. We can make this statement more explicit by computing $\Delta\sigma^0$. We take a Bondi system of coordinates with z the stereographic coordinate corresponding to $\hat{\mathbf{x}}$, and w_i corresponding to the momenta $\mathbf{k}_i$. In Minkowksi coordinates $\hat{\mathbf{x}}$ has components $P^{-1}(z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z})$. Note that a 1-form V_a given in Minkowski coordinates has z-component

$$V_z = \partial_z x^a \, V_a = \frac{(1-\bar{z}^2)V_1 - i(1+\bar{z}^2)V_2 - 2\bar{z}V_3}{P^2} \, r$$

when expressed in stereographic coordinates. We also want to find $\epsilon^m_{ab}(k_i) = \epsilon^m_a(k_i)\epsilon^m_b(k_i)$ such that $\epsilon^m(k_i)\cdot k_i = \epsilon^m(k_i)\cdot\epsilon^m(k_i) = 0$. We also choose a normalisation $\epsilon^+(k_i)\cdot\epsilon^-(k_i) = -1$. Then, we can take

$$\epsilon^{+\,a}(k^i) = \frac{1}{\sqrt{2}}(\bar{w}_i, 1, -i, -\bar{w}_i), \quad \text{and} \quad \epsilon^{-\,a}(k^i) = \frac{1}{\sqrt{2}}(w_i, 1, i, -w_i).$$

To compute $\Delta\bar{\sigma}^0$ we must find $\Delta^+ h_{zz}$. But we compute that

$$\epsilon^+_z(\omega\hat{\mathbf{x}}) = 0, \quad \text{and} \quad \epsilon^-_z(\omega\hat{\mathbf{x}}) = -\frac{\sqrt{2}r}{P}.$$

So the only contribution to $\Delta^+\bar{\sigma}^0$ is from $m=-$ (while $\Delta^+\sigma^0$ can be found from $m=+$). Moreover, we find that

$$k^i \cdot \epsilon^-(\hat{\mathbf{x}}) = \frac{\sqrt{2}(z-w_i)}{1+w_i\bar{w}_i}, \quad \text{and} \quad 1 - \hat{\mathbf{x}}\cdot\mathbf{k}^i = \frac{2(z-w_i)(\bar{z}-\bar{w}_i)}{(1+w_i\bar{w}_i)(1+z\bar{z})}.$$

Altogether, and using equation (24), we find

$$\Delta^+\bar{\sigma}^0 = 2G\int \mathrm{d}^2 w \left(\frac{z-w}{\bar{z}-\bar{w}}\,\frac{1+z\bar{z}}{1+w\bar{w}}\right)\left(\sum_i \eta^i \, E^i \, \delta^{(2)}(w-w_i)\right). \tag{25}$$

The expression in the first parenthesis is a Green's function for $\eth^2$. Acting on it with $\eth^2$ gives[13] $2\pi\delta^{(2)}(z-w)$. So

$$\eth^2\Delta^+\bar{\sigma}^0(z,\bar{z}) = 4\pi G\sum_i \eta^i E^i\delta(z-w_i).$$

In this way, one can infer the energies of the gravitons directly from the displacements of a sphere of masses: up to the limit imposed by the spacing of the masses[14]. We can use our earlier result, equation (7), to determine the change in Bondi mass that results from the passage of gravitons to $\mathscr{I}^+$. We find

$$\Delta M_B = \sum_k \eta_k E_k - \frac{1}{4\pi G}\int \mathrm{d}u \int \mathrm{d}\mu\, \dot{\sigma}^0\dot{\bar{\sigma}}^0.$$

By the conservation of energy, the sum vanishes. (Notice that our earlier statement that $\eth^2\dot{\bar{\sigma}}^0$ has no $l=0$ component has now become the conservation of energy.) Otherwise, we see that the only contribution to ΔM_B is due to the news and it is non-positive – in agreement with the mass loss formula. For the case we are considering, all the scattering products are massless gravitons and $M_B \to 0$ as $u \to \infty$[15]. We thus infer that the Bondi mass in the $u \to -\infty$ limit, is

$$M_B \to M_{ADM} = \frac{1}{4\pi G}\int \mathrm{d}u \int \mathrm{d}\mu\, \dot{\sigma}^0\dot{\bar{\sigma}}^0.$$

The computation for $\mathscr{I}^-$ shows that M_{ADM} is also the integrated news on $\mathscr{I}^-$. We thus infer that the integrated advanced news is the same as the integrated retarded news.

[13] See also equations (2.24)-(2.25) of ref. [5].

[14] It is reassuring to see that our determination of the energies in this way is Lorentz invariant (in the case of Minkowski spacetime). Lorentz transformations on Minkowski space induce global conformal transformations on the sphere at null infinity. Both sides of the equation have the same conformal weight: so that the E^i inferred from our equation are unaffected by Lorentz transformations.

[15] Indeed, Christodoulou and Klainerman found bounds on the rate of decay in ref. [2].

6 No Particle is Forgotten

Processes involving massive objects have long been known to emit long wavelength gravitational radiation that result in a memory effect. This result can be derived using linearised gravity, as in section 5, except that the energy-momentum tensor is now taken to have contributions from massive particles [30, 31]. However, as we discussed in section 4, the amplitudes for massive particles (of any spin) have the same soft graviton factorisation that we found for graviton amplitudes. Thus, the calculation that we did in §5.2 allows us to find the change in the asymptotic metric near $\mathscr{I}^+$ due to the scattering of massive particles. To make direct contact with the original formulas we omit the mild complication of using helicities and simply understand the RHS of the formula to be the transverse-traceless component of the expression. Then, for massive particles with momenta p^i we have

$$\Delta^+ h_{ab}(\hat{\mathbf{x}}) = \frac{4}{r}\sum_i \frac{\eta^i p^i_a p^i_b}{p^i \cdot x} = \frac{4}{r}\sum_i \frac{\eta^i m^i}{\sqrt{1-v_i^2}}\frac{v^i_a v^i_b}{1-|v^i|\cos\theta} + O(r^{-2}),$$

where $p^i_a = (1-v_i^2)^{-1/2} m^i v^i_a$. This agrees with the equations quoted in refs. [27, 30]. To find the proper displacements we compute $\Delta\sigma^0$ using stereographic coordinates. For some velocity $\mathbf{v}$ we then find a z-component $v_z = P^{-1} r v \sin\theta$, where θ is the angle between the point z on the sphere and the point on the sphere corresponding to $\mathbf{v}$. Then we find

$$\Delta^+ \sigma^0 = \sum_i \frac{2\eta^i m^i}{\sqrt{1-v_i^2}}\frac{v_i^2 \sin^2\theta_i}{1-v^i\cos\theta_i}. \tag{26}$$

This gives proper length displacements (via equation (14)) that agree with those derived through different methods in ref. [31]. As in the massless case, we would like to this result to the mass aspect. Unlike the massless case, we do not expect $M_B \to 0$ near i^+. We can compute the mass aspect of a free massive particle in the following way. The Schwarzschild metric in a centre of mass frame near $\mathscr{I}^+$ is $g = (-1 + 2M/r)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + ...$, from which we see that $\Psi = -2M$ in the COM frame. Suppose we have a tetrad for the Schwarzschild metric which is related to the COM frame by a boost. It is easy to infer what Ψ_2^0 is in this boosted frame. In fact, in order to preserve the Bondi gauge, one would have to perform both a boost and a null rotation around ι^α. However, under a null rotation around ι^A, Ψ_2 transforms to $\Psi_2 + 2\lambda\Psi_1 + \lambda^2\Psi_0$, for some λ. So the leading component Ψ_2^0 is not affected by a null rotation. We consider a boost of the form $n^a \mapsto K^{-1}n^a$ and $l^a \mapsto Kl^a$ where K is some combination of $l \leq 1$ harmonics. In particular, for a boost with 4-velocity $v^a = \gamma_v(1,0,0,v)$ we take[16] $K = \gamma_v(Y_0 + vY_1^0)$. Notice that since $l = \partial/\partial r$, the radial coordinate associated to the Bondi frame transforms as $r \mapsto r' = Kr$. Now, ψ_2 has boost weight 0. So, Ψ_2 is related to $\hat{\Psi}_2$ by

$$\Psi_2 = \hat{\Psi}_2 + O(r^{-4}) = \hat{\Psi}_2^0 r'^{-3} + O(r^{-4}) = \hat{\Psi}_2^0 \left(\frac{r}{r'}\right)^3 r^{-3} + O(r^{-4}).$$

But, in the COM, frame, $\hat{\Psi}_2^0 = -2M$. So

$$\Psi = -\frac{2M}{K^3} = -\frac{2M}{\gamma_v^3(1 + v\cos\theta)^3},$$

which agrees with the coordinate calculation performed in Appendix B of ref. [31]. Performing the elementary integral over the sphere gives $M_B = \gamma_v M$, which is what we expect for a massive particle. Moreover, starting from equation (20) one can infer that

$$\eth^2 \Delta^+ \bar{\sigma}^0 = 2\sum \eta^i M^i K_i^{-3},$$

where we assume that the contribution to $\Delta\bar{\sigma}^0$ is soft and so ignore $\dot{\sigma}^0\dot{\bar{\sigma}}^0$. This is the starting point for Ludvigsen's [15] calculation of the memory for massive particles. His result agrees with the formula, equation (26), that we found from the soft factor.

7 Discussion

We studied the leading order contribution to the memory effect near $\mathscr{I}^+$ and $\mathscr{I}^-$ and related it to Weinberg's soft factorisation of graviton amplitudes. We found that the leading order displacements can be used to infer scattering data. Moreover, we found that

[16] One can choose the harmonics to be normalised such that $\langle K, K\rangle = v^a v_a = 1$ where $\langle\ ,\ \rangle$ is the Lorentzian inner product on the $l \leq 1$ harmonics [15].

the memory effect near a generator of $\mathscr{I}^+$ is the same as for the corresponding generator of $\mathscr{I}^-$: where the null generators are matched at i^0. This is not an obvious result. From our calculation, we see that it is a consequence of the soft factorisation of graviton amplitudes. This, in turn, can be regarded as the Ward identity for diagonal supertranslations. Soft gravitons could be regarded as the Goldstone bosons of this symmetry (which is a symmetry of the S-matrix but not of the interior spacetime). Hence, the memory effected by soft gravitons ought to be equivalent to the displacements induced by the action of some diagonal supertranslation. By this reasoning, the memory effect should indeed be the same on diagonally identified generators. This appears to explain our result. We also remarked that the integrated news is the same for $\mathscr{I}^+$ and $\mathscr{I}^-$. This does not appear as a consequence of the diagonal supertranslation symmetry, but as a consequence of energy conservation (which corresponds to a translation subgroup of the diagonal supertranslations). Finally, let us comment on loop corrections. The leading order memory effect that we have calculated is not affected by loop corrections since it involves only the leading term in the soft expansion of the amplitudes (which is tree-level exact). However this would no longer be the only contribution if we considered geodesic deviation at a finite-radius or for a finite-time. To compute the memory in this case would involve integrating over soft gravitons with a range of energies near $\omega = 0$. (Some remarks on finite-radius, finite-time conservation laws recently appeared in appendix D of ref. [32].) In contrast with §5.2, in which we took the $\omega \to 0$ limit, such a calculation would involve the subleading terms that appear in the soft expansions of the amplitudes. But these are sensitive to loop corrections (as recently discussed in ref. [23]). This calculation does not appear to have been done.

A Peeling Theorems

Let o^A, ι^A be the spin frame corresponding to the metric g_{ab} employed in §2. Then we can construct a spin frame for $\hat{g}_{ab} = \Omega^2 g_{ab}$ by choosing, $\hat{o}^A = \Omega^{-1} o^A, \hat{o}_A = o_A$. One can verify that $\hat{o}^A$ is parallel transported with respect to $\hat{l}^a = \hat{o}^A \hat{\bar{o}}^{\dot{A}}$. However, ι^A is not. So we define $\hat{\iota}^A$ so that $\hat{\iota}^A \approx \iota^A$ and $\hat{\iota}^A$ is parallel transported with respect to $\hat{l}^a$. Now, we would like to write $\hat{N}_a$ in terms of the tetrad components. To this end, introduce an affine parameter $\hat{r}$ such that $\hat{r} \approx 0$ and $\hat{n}_a = \partial_a \hat{r}$. Notice that $\hat{r}$ is increasing along the null geodesics generated by l^a since $l^a \partial_a \hat{r} = 1$. So, we can expand Ω along one of these null geodesics:

$$\Omega = \Omega_1 \, \hat{r} + \Omega_2 \, \hat{r}^2 + \Omega_3 \, \hat{r}^3 + O(\hat{r}^4).$$

We compute,

$$\frac{\mathrm{d}^2\Omega}{\mathrm{d}\hat{r}^2} = (\hat{l}^a \hat{\nabla}_a \hat{l}^b)\hat{\nabla}_b \Omega + \hat{l}^a \hat{l}^b \hat{\nabla}_a \hat{\nabla}_b \Omega = \hat{l}^a \hat{l}^b \hat{\nabla}_a \hat{\nabla}_b \Omega,$$

where we have used the geodesic equation. However, it follows from equation (1) together with the $C^{>2}$ smoothness of $\hat{g}_{ab}$ and Ω that $\hat{l}^a \hat{l}^b \hat{\nabla}_a \hat{\nabla}_b \Omega \approx 0$. So the expansion becomes

$$\Omega = \Omega_1 \hat{r} + \Omega_3 \, \hat{r}^3 + O(\hat{r}^4). \tag{27}$$

Taking derivatives gives

$$\hat{N}_a = (\Omega_1 + O(\Omega^2))\hat{n}_a + O(\Omega^3),$$

which follows from the definitions of $\hat{N}_a$ and $\hat{n}_a$ in terms of Ω and $\hat{r}$. In §2.3 we adopt a frame satisfying $\hat{l}^a \nabla_a \Omega \approx -1$. For this frame one has $\Omega_1 = -1$. We can expand $\hat{r}$ and Ω in terms of the affine parameter r for the l^a geodesics. Indeed, since $\hat{l}^a = \Omega^{-2} l^a$ we have $\mathrm{d}\hat{r}/\mathrm{d}r = \Omega^2$. Using equation (27) we can perform this integration to find

$$r = -\Omega_1^{-2} \hat{r}^{-1} + \mathcal{B}_0 + \mathcal{B}_1 \hat{r} + O(\hat{r}^2),$$

for some smooth $\mathcal{B}_i$. We can invert this expansion for $\hat{r}$ to find

$$\hat{r} = -\Omega_1^{-2} r^{-1} + \mathcal{C}_2 r^{-2} + O(r^{-3}), \quad \text{and} \quad \Omega = \Omega_1^{-1} r^{-1} + \mathcal{D}_2 r^{-2} + O(r^{-3}).$$

Consider now a conformal density $\mathcal{A}$ of weight $-w$. Then, under a conformal transformation $g_{ab} \mapsto \Omega \hat{g}_{ab}$ we have $\mathcal{A} \mapsto \hat{\mathcal{A}} = \Omega^{-w}\mathcal{A}$. We assume that $\hat{\mathcal{A}}$ is $C^{>2}$ so that

$$\hat{\mathcal{A}} = \hat{\mathcal{A}}_0 + \hat{\mathcal{A}}_1 \hat{r} + \hat{\mathcal{A}}_2 \hat{r}^2 + O(\hat{r}^3).$$

Then, using the large r expansions for $\hat{r}$ and Ω, we find

$$\mathcal{A} = \Omega_1^{-w} \hat{\mathcal{A}}_0 r^{-w} + \mathcal{A}_1 r^{-w-1} + O(r^{-w-2}).$$

This is the peeling theorem for scalar densities. We find a similar result for tensor densities. Let $T_{A...}$ be a tensor density of weight $-w$ and let

$$\mathcal{A} = T_{A...B...\dot{A}...\dot{B}...} \underbrace{o^A ... \bar{o}^{\dot{A}} ...}_{p} \iota^B ... \bar{\iota}^{\dot{B}} ...$$

be a component with p o or $\bar{o}$ indices. We have $\hat{o}^A = \Omega^{-1} o^A$ and one can show that $\hat{\iota}^A = \iota^A + O(\Omega)$. Given this it follows that

$$\mathcal{A} = \Omega^{w+p} \hat{T}_{A...B...\dot{A}...\dot{B}...} \underbrace{\hat{o}^A ... \hat{\bar{o}}^{\dot{A}} ...}_{p} (\hat{\iota}^B + O(\Omega)) ... (\hat{\bar{\iota}}^{\dot{B}} + O(\Omega)) ...$$

So, assuming $\hat{T}$ is $C^{>2}$ near $\mathscr{I}$, we find an expansion

$$\mathcal{A} = \mathcal{A}_0 r^{-w-p} + \mathcal{A}_1 r^{-w-p-1} + O(r^{-w-p-2}).$$

B Stationary Phase

In the text we claimed that integrals of the form

$$I = \int_A \mathrm{d}x \phi(x) e^{irf(x)}$$

only receive contributions from neighbourhoods of stationary points satisfying $df = 0$. To see this, consider the same integral over an open subset $B \subset A$ such that $df \neq 0$ on B. Then, if ϕ is a C^k function we can perform integration by parts k times to find that the integral is $O(r^{-k})$. So we restrict ourselves to small neighbourhoods of stationary points. On these neighbourhoods we take normal coordinates y_i such that $f = f_* + (y_1^2 + ... + y_s^2 - y_{s+1}^2 - ... - y_n^2)/2$. Notice that

$$\mathrm{diag}(1, \dots, -1, \dots) = \frac{\partial^2 f}{\partial y_i \partial y_j} = \frac{\partial x_m}{\partial y_i} \frac{\partial^2 f}{\partial x_m \partial x_n} \frac{\partial x_n}{\partial y_j}.$$

So $|\det(\partial y/\partial x)| = H^{-1/2}$, where H is the determinant of the Hessian of f in the original coordinates. Thus, the integral over the neighbourhood around one critical point takes the form

$$I_* = \frac{e^{irf_*}}{\sqrt{H}} \int \mathrm{d}y \phi(y) e^{\frac{ir}{2}\left(y_1^2 + ... + y_s^2 - y_{s+1}^2 - ... - y_n^2\right)}.$$

If we Taylor expand $\phi(y) = \phi(0) + y_i \partial_i \phi(0) + O(y^2)$ we can perform integration by parts k times on the $O(y^k)$ terms to find that these are $O(r^{-k})$. The leading term involving $\phi(0)$ is just a product of Gaussian integrals. Performing these integrals we conclude that

$$I_* = \left(\frac{2\pi}{r}\right)^{n/2} e^{\frac{i\pi}{4}(n-2s)} \frac{e^{irf_*}}{\sqrt{H_*}} \phi_*. \tag{28}$$

Notice that $2s - n$ is just the signature of the Hessian evaluated at the critical point. This result is the stationary phase formula. For further remarks on this result see ref. [33]. Finally, we consider $\int \mathrm{d}\omega \exp(-2i\omega r)$. The partial integral is

$$I_a = \int_0^a \mathrm{d}\omega e^{-2i\omega r} = \frac{i}{2r}\left[e^{-2iar} - 1\right] = -\frac{i}{r} e^{-iar} \sin(ar).$$

We see that $|I_a| \leq r^{-1}$. This bound doesn't depend on a, and so $\int \mathrm{d}\omega \exp(-2i\omega r) = O(r^{-1})$. This is used, along with the stationary phase formula, in section 5.

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