Notes on static spacetimes

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Abstract

Several popular spacetimes (Schwarzschild, Curzon, the C-metric) are static. These notes consider static spacetimes in general. By making this restriction, we free ourselves from the full Einstein equations and study a simpler set of equations. This simplification leads us to an unambiguous boundary value problem for static spacetimes. (This is an observation due to M. Anderson.) I compare this to the usual situation. I also discuss optics and the definition of mass for static spacetimes, giving examples. This material was prepared for a series of seminars at the ENS de Lyon (April 2017). The interest of my audience was my computation in section 4.2, which may find application in cosmological investigations concerning the density of images on a sky—but I don't have the expertise to elaborate this.

1 Prologue: the local field equations

The local field equations of general relativity, $G_{ab} = 8\pi G T_{ab}$, can be recast as a statement relating energy density at a point to the volume of balls around that point. If U^a is the 4-velocity of an observer, the energy density observed by the observer is $T(U, U)$. On the other hand, a geometrical calculation shows that the quantity $G(U, U) = R(U, U) + R/2$ controls the volume to area ratio (the 'isoperimetric ratio') of small balls in the spacelike neighbourhood of U. So we may interpret $G(U, U) = 8\pi G T(U, U)$ as a statement relating local energy density at a point to the distortion of small balls around that point. The precise relation is equation (1), which we will now derive. To do this we will need Cartan's formula relating the metric to the curvature. We begin by deriving Cartan's formula. The techniques used in the derivation will be useful later in the essay, in section 4.2. The key idea is to use geodesic coordinates in the neighbourhood of a point. Let's recall the construction of geodesic coordinates, which is discussed in all Riemannian geometry textbooks, such as Petersen's [1]. If M is Riemannian and $p \in M$ is a point, there is a local correspondence between T_pM and the neighbourhood of p. This is given by the exponential map

$$\exp : T_pM \to M : k \mapsto x_k(1),$$

which maps the vector $k \in T_pM$ to the point $x_k(1) \in M$, where $x_k(t)$ is the geodesic with $x_k(0) = p$ and $\dot{x}_k(0) = k$. The usual coordinates on T_pM are then called the 'geodesic coordinates' for M in the neighbourhood of p. We can equip T_pM with the usual Euclidean metric,

$$e : T_kT_pM \times T_kT_pM \to \mathbb{R}.$$

At a point $k \in T_pM$, the derivative of exp should give us a map

$$T_k \exp : T_kT_pM \to T_{x_k(1)}M.$$

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Then the metric e will push forward to a metric on M. To construct this map, fix some tangent vector $s \in T_k T_p M$. We can regard s as being the tangent to the line $k(\lambda) = k + \lambda s$, for some parameter λ, at $\lambda = 0$. This line in $T_p M$ exponentiates to a family of geodesics in M, $x_{k+\lambda s}(t)$. Then $T_k \exp$ is the map

$$T_k \exp : s \mapsto \frac{\partial}{\partial \lambda} x_{k+\lambda s}(1) \bigg|_{\lambda=0}.$$

We would like to present $T_k \exp$ explicitly. We can do this when $||k||$ is small using a Taylor series expansion. In effect, we are studying the Jacobi field

$$v(t) = \frac{\partial}{\partial \lambda} x_{k+\lambda s}(t) \bigg|_{\lambda=0}.$$

By the definition of $x_k(t)$ we immediately conclude that

$$v(0) = 0 \qquad \text{and} \qquad \dot{v}(0) = s.$$

For small t, we can Taylor expand,

$$v(t) = v(0) + \dot{v}(0)t + \frac{1}{2}\ddot{v}(0)t^2 + \frac{1}{3!}\dddot{v}(0)t^3 + ...$$

We use Jacobi's equation to find[1]

$$\ddot{v}(0) = 0, \qquad \text{and} \qquad \dddot{v}(0) = -R(k, s)k.$$

In sum, we have the following expansion,

$$T_k \exp : s \mapsto s - \frac{1}{3!} R(k, s)k + ...,$$

where the ellipsis suppresses terms of higher order in k. Alternatively, we could replace k with $r\hat{k}$ where $||\hat{k}|| = 1$, and then, in geodesic coordinates,

$$T_k \exp : s^i \mapsto s^i - \frac{r^2}{3!} R^i{}_{jkl} \hat{k}^j s^k \hat{k}^l + \mathcal{O}(r^3).$$

The pushforward of the metric e is then

$$(T_k \exp)_* \, e(s, t) = \delta_{ij} \left(s^i - \frac{r^2}{3!} R^i{}_{jkl} \hat{k}^j s^k \hat{k}^l + \mathcal{O}(r^3) \right) \left(t^i - \frac{r^2}{3!} R^i{}_{jkl} \hat{k}^j t^k \hat{k}^l + \mathcal{O}(r^3) \right).$$

We can write this in component form as

$$g_{ij} = \delta_{ij} - \frac{1}{3} r^2 R_{imjn} \hat{k}^m \hat{k}^n + \mathcal{O}(r^3).$$

This is Cartan's formula for the metric near p given in local geodesic coordinates.[2] Working in D dimensions, the associated volume form near p is

$$e^{(D)} = \left(1 - \frac{1}{3} r^2 R_{imin} \hat{k}^m \hat{k}^n + \mathcal{O}(r^3) \right) r^{D-1} \mathrm{d}r \wedge \mathrm{d}\Omega^{D-1},$$

[1] Recall that $\nabla_k v = \nabla_v k$, so that

$$\ddot{v} = R(k, v)k.$$

This is Jacobi's equation. Differentiating we find

$$\dddot{v} = R(k, \dot{v})k.$$

The result follows setting $t = 0$.

[2] Our derivation is an elaboration of that which appears in Woodhouse's textbook, ref. [2]—see Chapter 9, Footnote 11.

where $\hat{k}$ is not regarded as a set of spherical coordinates and $\mathrm{d}\Omega^{D-1}$ is the usual volume form on the unit sphere. The key relation that we need is

$$\int \mathrm{d}\Omega^{D-1}\hat{k}^i\hat{k}^j = \frac{1}{D}\delta^{ij}\Omega^{D-1},$$

where Ω^{D-1} is the volume of the unit sphere. Then consider the ball $r \leq r_*$. It has volume

$$V(r_*) = \int e^{(D)} = \Omega^{D-1}\int_0^{r_*} \mathrm{d}r \left(1 - \frac{1}{3D}r^2 R_{ijij} + ...\right) r^{D-1}.$$

On the other hand, the boundary of the ball has area

$$A(r_*) = \Omega^{D-1}\left(1 - \frac{1}{3D}r_*^2 R_{ijij} + ...\right) r_*^{D-1}.$$

So a short computation shows that the isoperimetric ratio between the area and volume is

$$\frac{A(r)}{V(r)} = \frac{D}{r}\left(1 - \frac{2}{3D(D+2)}r^2 R_{ijij} + ...\right).$$

The first term is the relation as it holds in flat space. The second term is a correction, which is related to the energy density at the point p. We can complete T_pM to be the tangent space to a Lorentzian manifold by adding the observer's velocity U as a basis vector for the time-like direction. Then, denoting the U direction by the index 0, we have

$$R_{ijij} = R_{ii} + R_{i0i0} = R + 2R_{00},$$

where we use, for instance, that $R_{0000} = 0$ by the symmetries of the Riemann tensor. Einstein's equation is then the statement that, near p, the observer associated to U would experience isoperimetric ratios of the form

$$\frac{A(r)}{V(r)} = \frac{D}{r}\left(1 - \frac{4}{3D(D+2)}r^2 T_{00} + ...\right). \tag{1}$$

The full field equations assert that this holds for all observers. We will not make any use of this relation in the essay. However, geodesic coordinates and the exponential map play an essential role in section 4.2, where we use this construction to compute a Jacobian which describes the lensing on the sky of an observer in a static spacetime.

2 The boundary value problem

We will call a spacetime *static* if it is topologically of the form $M \times \mathbb{R}$ (or $M \times S^1$) and is equipped with a product metric. Our aim in this section is to formulate the problem of finding static spacetimes satisfying Einstein's equations as a boundary value problem. The reason that this is interesting is that general relativity is not a well-posed boundary value problem. That is, the boundary data for general relativity cannot be specified freely. Before we consider static spacetimes, we will recall some facts about general relativity. Let M, g be a manifold with boundary. Then the Einstein-Hilbert action is

$$S_{EH} = \int_M eR[g],$$

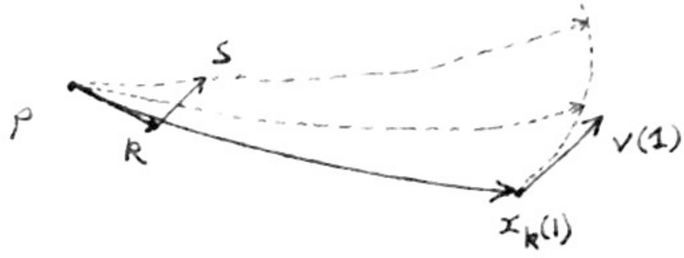

Figure 1: Finding the image of s under $T_k \exp$ via Jacobi fields. $x_k(1)$ is the image under exp of $k \in T_pM$. In this image, $T_k \exp s = v(1)$.

where e is the volume form for g. The Einstein-Hilbert action does not give a well defined variational principle for Dirichlet boundary data at ∂M. This is because the variation of S has boundary terms which do not vanish for Dirichlet boundary conditions. To surmount this problem, Gibbons and Hawking [3] proposed the following action

$$S_{GH} = \int\limits_M eR + 2 \int\limits_{\partial M} \hat{e}K.$$

We have introduced $\hat{e}$ for the induced volume form on ∂M. This action does give a well defined variational principle for the Dirichlet problem. To see that this proposal does work out, let h be a perturbation of g. The first order change in R is

$$\delta R = -\nabla_a \nabla^a (g^{ab} h_{ab}) + \nabla^a \nabla^b h_{ab} - R_{ab} h^{ab}.$$

and in the volume form it is

$$\delta e = \frac{1}{2} g^{ab} h_{ab} e.$$

So the first order variation of eR gives a boundary term

$$\delta S_{EH} = ... - \int\limits_{\partial M} \hat{e} \left(n(g^{ab} h_{ab}) + n^a \nabla^b h_{ab} \right),$$

for outward normal vector n. The difficulty is the first term, which is a normal derivative of h. We do not keep this fixed under Dirichlet boundary conditions. It is necessary then, to add a boundary term whose variation cancels this normal derivative. Recall that the trace of the extrinsic curvature is

$$K = \frac{1}{2} \pounds_n \det g.$$

From this we find the first variation

$$\delta K = \frac{1}{2} n(g^{ab} h_{ab}).$$

This explains the Gibbons Hawking proposal. In sum, the variation of the Gibbons-Hawking action has the boundary term

$$\delta S_{GH} = ... + \int\limits_{\partial M} \hat{e} h_{ab} K^{ab},$$

which clearly vanishes for for Dirichlet variations of the metric (for which the metric variation h_{ab} is set to zero at the boundary). This appears to show that the Einstein-Hilbert action, suitably modified, gives a Dirichlet problem for Einstein metrics. However, the Dirichlet data, $i^* g_{ab}$, cannot be freely specified and so Einstein's equation does not correspond to a true Dirichlet problem. The constraints on the boundary data are G_{00} and G_{0i} where the '0' index corresponds to the direction which is normal to the boundary. We will now consider the constraint G_{00} in detail. A useful presentation of G_{00} is as follows. Consider the normal neighbourhood $N(\partial M)$ of ∂M and let t be a parameter in the normal direction so that ∂M is the zero section. That is, t defines a foliation of the neighbourhood with some normalised normal vector n,

$$g(n,n) = \pm 1,$$

and induced metrics on the $t =$ constant leaves,

$$\hat{g}_{ab} = g_{ab} \mp n_a n_b.$$

The curvatures are related by

$$R = \hat{R} \pm (K^2 - K_{ij}K^{ij}) \pm 2\nabla_i (n^j \nabla_j n^i - n^i \nabla_j n^j), \tag{2}$$

so that the Einstein-Hilbert action on $N(\partial M)$ decomposes as

$$\int eR = \int \sqrt{g^{00}} \mathrm{d}t \wedge \hat{e} \left(\hat{R} \pm (K^2 - K_{ij}K^{ij}) \right).$$

A Dirichlet variation of g^{00} then yields the constraint

$$G_0 0 : \qquad \hat{R} \pm (K^2 - K_{ij}K^{ij}) = 0$$

on the induced metric $\hat{g}_{ab}$. (Note that the Euler-Lagrange equation corresponding to the variation of g_{00} is precisely the normal-normal component, G_{00}, of the Einstein tensor.) This is a bona fide constraint: the intrinsic curvature $\hat{R}$ contains second derivatives of $\hat{g}_{ab}$ while the extrinsic curvatures contain first derivatives. So this constraint is not satisfied for generic $\hat{g}_{ab}$. The boundary data must also satisfy the other equations of motion: $G_{0i} = 0$. Methods to produce valid boundary data for general relativity were first studied by Lichnerowicz [4]. Significant elaborations were later made by York—see [5] and the textbook [6].

2.1 The static boundary value problem

We take a static spacetime to be topologically a product manifold $\mathbb{R} \times M$, for some Riemannian M, with a metric

$$\hat{g} = -u^2 \mathrm{d}t^2 + g.$$

Here, $\mathrm{d}t$ is a basis 1-form on $\mathbb{R}$ and u is a function on M. The function u is often called the 'lapse function'. Both u and the Riemannian metric g are independent of t. Since the $t =$ constant leaves have no extrinsic curvature, equation (2) gives

$$S = \int_{\mathbb{R}\times M} \hat{e}\hat{R} = \int \mathrm{d}t \wedge euR.$$

If we 'compactify' $\mathbb{R}$ by replacing it with a circle, the action becomes

$$S = \beta \int_M euR,$$

for some constant β. Take $\beta = 1$. For a variation of u we find the Euler-Lagrange equation

$$R = 0.$$

For a metric perturbation h of g, we find the variation

$$\delta S = \int\limits_M eu \left(-\triangle(h^i_i) + \nabla^i\nabla^j h_{ij} - R_{ij}h^{ij} + \frac{1}{2}Rh^i_i \right).$$

The bulk term in δS (after integrating twice by parts) gives the Euler-Lagrange equation

$$uR_{ab} - \nabla_a\nabla_b u + g_{ab}\triangle u = 0.$$

Taking the trace of this and using $R = 0$ we infer that the lapse function is harmonic,

$$\triangle u = 0.$$

The Euler-Lagrange equations are, therefore,

$$\triangle u = 0, \qquad \text{and} \qquad uR_{ab} = \nabla_a\nabla_b u. \tag{3}$$

Two integrations by parts were used to obtain the bulk term in δS. These give rise to a boundary term given by

$$\delta S = ... + \int\limits_{\partial M} \star \left(uK^{ab}h_{ab} - g^{ab}h_{ab}\, n(u) + 2u\delta K \right).$$

This is interesting because it suggests a new approach to the boundary value problem. If we fix g_{ab} and K at the boundary ∂M then δS has no boundary term. That is, the Einstein-Hilbert action, restricted to static metrics, gives a well defined variational problem for the mixed boundary data (g_{ab}, K). In our discussion of the Einstein-Hilbert action, the Dirichlet boundary data was constrained. Constraints do not arise in the present case. Let γ be the induced metric on one of the boundary components of M. Let K_{ij} be the extrinsic curvature and n^i be the normal vector for this boundary. The contracted Codazzi relation—equation (2)—then reads

$$R[g] = 0 = R[\gamma] + 2R_{ab}[g]n^a n^b + K_{ij}K^{ij} - K^2.$$

Now, using the Euler-Lagrange equation we find that

$$\begin{aligned} R_{ab}n^a n^b &= u^{-1} n^a n^b \nabla_a\nabla_b u \\ &= u^{-1}\nabla_n\nabla_n u - u^{-1}K^{ab}n_a\nabla_b u. \end{aligned}$$

So, while the first and second normal derivatives of γ appear in the constraint, so do the first and second derivatives of u. Since u is not fixed by the boundary data, we can choose it to satisfy the constraint for any freely prescribed (γ, K). Static gravity is thus a well posed boundary value problem on an $n-1$-manifold with mixed (von Neumann-Dirichlet) boundary conditions on the $n-2$-dimension boundary components. The only author whom I have found to emphasize this point in Anderson, see [7, 8].

2.2 Internal horizons

Suppose we have a static spacetime M, g, u, as above. Define an *internal horizon* as any connected component of the vanishing set of u. We assume that horizons have codimension 1. Let H be a horizon. Then the static Einstein equations give

$$\nabla_a\nabla_b u|_H = 0.$$

This means, in particular, that $n(u)$ is constant on H. Also, for any two vectors X, Y tangent to H we have

$$0 = X^a Y^b \nabla_a\nabla_b u = X^a Y^b K_{ab}\, n(u).$$

Now, $n(u)$ cannot vanish, lest u be everywhere zero: this is follows from the usual uniqueness theorem for the Laplacian. So we must have that the extrinsic curvature K_{ab} vanishes on the horizon. In particular, the trace, K, vanishes: which means that H is a minimal surface with respect to g. Finally, we can identify $n(u)$ as the surface gravity of the horizon, κ. To make this identification we must recall the spacetime interpretation of a static solution. Here,

$$K = \frac{\partial}{\partial t}$$

is a Killing vector and $K^2 = -u^2$. Then the ordinary spacetime formula for the surface gravity is[3]

$$\kappa = \lim_{u \to 0} \sqrt{-A^2 K^2},$$

where A^a is the acceleration $A^a = U^b \nabla_b U^a$ associated to a static observer $U = u^{-1} K$. A standard computation gives

$$A^a = \nabla^a \log u,$$

where $\log u = \log(-K^2)/2$. Moreover, since u is constant on the horizon, we see that

$$\nabla^a \log u = n(\log u) n^a,$$

near the horizon. This gives us the identification $\kappa = n(u)$.

2.3 Conformally flat examples

In what remains of this section, we would like to find several examples of static spacetime that we can refer back to in sections 3 and 4. We begin by looking for solutions M, g, u where g is conformally flat. Taking the ansatze

$$g_{ab} = e^{2f} \delta_{ab},$$

the Ricci curvature is[4]

$$R_{ab} = -(n-2)\left(\partial_a \partial_b f - \partial_a f \partial_b f\right) - \left(\partial^2 f + (n-2)||\partial f||^2\right)\delta_{ab}.$$

On the other hand, the lapse equation, $\triangle u = 0$, can be expanded as

$$\partial_a \partial_b u + (n-2)\partial_a f \partial_b u = 0.$$

This can be suggestively re-written as

$$\partial^2(\psi u) - u\partial^2 \psi = 0, \qquad \text{where} \quad \psi = e^{\frac{n-2}{2} f}.$$

Indeed, in terms of ψ, the Ricci flat condition, $R = 0$, becomes

$$\partial^2 \psi = 0.$$

Together, the lapse equation and the Ricci flat condition are thus reduced to the demand that ψ and ψu are harmonic functions. So, Einstein's equations are reduced to

$$\partial^2 \psi = 0, \qquad \partial^2(\psi u) = 0, \quad \text{and} \quad u R_{ab} = \nabla_a \partial_b u. \tag{4}$$

Recalling our derivation of the static Einstein equations, we see that the first two of these equations are sufficient to solve the constraints $G_{0,\mu}$ on initial data. This means that 'momentarily static' initial data can be obtained simply by choosing two harmonic functions. If this initial data also solves

$$u R_{ab} = \nabla_a \partial_b u,$$

then it corresponds to a static solution. Something similar to what we have described here is used in the numerical relativity community to derive 'momentarily static' initial data for general relativity. See [11] for an example. The general method is outlined in detail in chapter 9 of [6].

[3] See section 2.5 of [9].

[4] For helpful collections of formulas of this kind, I recommend [10].

2.3.1 Schwarzschild

If we remove just one point from $M = \mathbb{R}^3$ we can take two nontrivial harmonic functions

$$\psi = 1 + \frac{a}{r}, \qquad \text{and} \qquad \phi = 1 + \frac{b}{r},$$

for some constants a and b. Setting the lapse $u = \phi/\psi$, our proposed static metric is then

$$\tilde{g} = -\left(\frac{1+\frac{b}{r}}{1+\frac{a}{r}}\right)^2 \mathrm{d}t^2 + \left(1+\frac{a}{r}\right)^2 \left(\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2\right).$$

However, in order for this to be a solution (and not merely static initial data), we must impose the Einstein equation

$$uR_{ab} = \nabla_a \partial_b u.$$

The rr component of this equation evaluates to give

$$-\phi\psi'' - \psi\phi'' + 4\phi'\psi' = 0,$$

where primes are r derivatives. Substituting our two solutions we find the condition

$$a + b = 0.$$

In this way we recover the Schwarzschild solution in isotropic coordinates—identifying a with the Schwarzschild radius r_S. Then $b = -r_S$ and we see that there is one horizon at radius $r = r_S$. In section 2.1 we saw that the static equations are a boundary value problem for the data (γ_{ij}, K), where γ is the boundary metric and K is the trace of the boundary's extrinsic curvature. In the present instance, γ is the sphere metric on both boundaries. What is K in our example? The trace of the extrinsic curvature of a constant radius sphere is

$$K(r) = \frac{1}{2}\pounds_n \det\gamma = \psi^2 r + 2\psi\partial_r\psi,$$

which is explicitly given by

$$K(r) = r + 2a + \frac{a^2}{r} - \frac{2a}{r^2} - \frac{2a^2}{r^3}.$$

So $K \sim r$ for large r and $K(a) = 0$. This is what we expect because, as shown earlier, K_{ij} vanishes on horizons. We also have $K \sim -2a^2/r^3$ for small r. We could think of this as defining two BVP's: one on the region $a \leq r < \infty$, the other on the region $0 < r \leq a$. Notice that

$$K(a) = \psi^2 r + 2\psi\partial_r\psi\big|_{r=a} = 0$$

is a mixed boundary condition for the Laplace equation $\partial^2\psi = 0$. On the other hand, u has Dirichlet boundary conditions. Namely, $u \to 1$ for large r, $u = 0$ for $r = a$, and $u \to -1$ near the puncture.

2.3.2 Multiple black holes

We can obtain static initial data for spacetimes with several black holes using the conformally flat ansatze. Somewhat trivially, we take

$$\psi = 1 + \sum \frac{a_i}{|x - x_i|}, \qquad \text{and} \qquad \phi = 1 + \sum \frac{b_i}{|x - x_i|}.$$

Near the i^{th} puncture, $u \to b_i/a_i$. Choosing the normalisation -1 gives $b_i = -a_i$. This is similar to our normalisation $+1$ for large r, and the horizon interiors can be considered asymptotic regions of their own. There is a resemblance to the Euclidean instanton solution [12]

$$\tilde{g} = \psi^{-1}(\mathrm{d}\tau + \theta)^2 + \psi \mathrm{d}x^i \mathrm{d}x^i.$$

Here θ is a 1-form given by $\omega = \mathrm{d}\theta = \star_3 \mathrm{d}\psi$. Since $\mathrm{d} \star_3 \mathrm{d}\psi = 0$, ω belongs to the relative de Rham group $H^2_{dR}(M)$. Given the singularities at the points x_i, the solution $\tilde{g}$ is not asymptotic to a Euclidean 4-sphere, as is discussed by [13]. We will encounter ω again when we define the mass of a region of static spacetime.

2.4 Axially symmetric examples

In this section we give examples for which our static spacetime $S^1 \times M$ has an axial killing vector in addition to being static. An interesting case is four dimensions, and this was investigated by H. Weyl in [14]: here there are as many cyclic coordinates (t and an angle ϕ) as there are non-cyclic. Weyl's metric then describes a family of tori over a half-plane: it could be regarded as a kind of integrable system.[5] It is possible to put the 3-metric in the following form[6]

$$g = u^{-2}\left(h^2(\mathrm{d}z^2 + \mathrm{d}r^2) + r^2 \mathrm{d}\phi^2\right), \tag{5}$$

where u is the lapse (independent of ϕ) and h is a conformal factor. To find the equations that h and u must satisfy, we apply our two Euler-Lagrange equations to this ansatze. The first equation is $\triangle_g u = 0$. The Laplacian associated to g is

$$\triangle_g = \frac{u^3}{h^2 r}\left[\partial_z\left(\frac{r}{u}\partial_z\right) + \partial_r\left(\frac{r}{u}\partial_r\right) + \partial_\phi\left(\frac{h^2}{ur}\partial_\phi\right)\right].$$

The first equation of motion is thus

$$\partial_z^2 \log u + \partial_r^2 \log u + \frac{1}{r}\partial_r \log u = 0.$$

In other words, $\log u$ is an axially symmetric harmonic function in Euclidean 3-space. Since $\log u$ plays such an important role, we will define $\mu = \log u$. Notice that Laplace's equation can also be written as

$$\partial_z(r\mu_z) + \partial_r(r\mu_r) = 0. \tag{6}$$

The second equation of motion is $uR_{ab} = \nabla_a \partial_b u$. Manipulating the zz and rr components gives

$$\nu_z z + \nu_r r = -(\mu_r^2 + \mu_z^2) \qquad \text{and} \qquad \nu_z \mu_z + \nu_r \mu_r = r\mu_r(\mu_z^2 + \mu_r^2),$$

which may be solved by

$$\nu_z = 2r\mu_z\mu_r, \qquad \text{and} \qquad \nu_r = r\mu_r^2 - r\mu_z^2. \tag{7}$$

In fact we only really need to take one of these equations. For instance, consider $\nu_z = 2r\mu_z\mu_r$. Since μ is harmonic, we find that

$$\nu_{zr} = 2\mu_{zr}(r\mu_r) - 2\mu_z\partial_z(r\mu_z) = \partial_z(r\mu_r^2 - r\mu_z^2).$$

So, if it is possible to integrate for ν, we will necessarily have the second equation, $\nu_r = r\mu_r^2 - r\mu_z^2$. Equations (7) and (6) constitute the equations of motion. We now give four interesting examples of solutions to these equations.

[5] More generally, we could consider a product metric on $T^k \times \mathbb{R}^k$ given by $\tilde{g} = h(x) + g$, where h is a metric on T^k whose components depend on the position $x \in \mathbb{R}^k$ and we take g to be conformally flat. We postpone discussion of this generalisation, and focus on four dimensions with one axial killing vector. In four dimensions, axially symmetric metrics are related to a class of self-dual $SU(2)$ Yang-Mills bundles; see [Wald] [Witten].

[6] It follows from the fact that any 2-metric can be written locally as a conformally flat metric.

2.4.1 Infinite line

Perhaps the easiest solution to imagine is the green's function for the Laplacian in two dimensions,

$$\mu = 2a \log r.$$

Integrating the equations of motion, we find

$$\nu = 4a^2 \log r + b,$$

for constants a and b. The associated metric is

$$g = -r^{4a}\mathrm{d}t^2 + c^2 r^{8a^2-4a}(\mathrm{d}z^2 + \mathrm{d}r^2) + r^{2-4a}\mathrm{d}\phi^2.$$

For $a = 0$, g is Minkowski, while for $a = 1/2$, it is merely a patch of Minkowski space. So we do not interpret g as the metric for an infinite line singularity for all values of a. We might regard $a = 1/4$ as a limiting case, since the circular geodesics $z, r =$ constant become null at this value of a. [15]

2.4.2 Curzon

The fundamental solution to Laplace's equation in cylindrical coordinates is

$$\mu = -\frac{1}{\sqrt{(z-z_o)^2 + r^2}}.$$

This Coulomb-like potential would describe a point particle in classical theory. Here, it gives rise to the Curzon metric, which is not the Schwarzschild solution. Choose $z_o = 0$ and write $F = \sqrt{(z-z_o)^2 + r^2}$. Then

$$\mu_z = \frac{z}{F^3}, \qquad \text{and} \qquad \mu_r = \frac{r}{F^3}.$$

We are left to integrate

$$\nu_z = \frac{2r^2 z}{F^6} = -\frac{1}{2}\partial_z\left(r^2 F^{-4}\right).$$

Introducing a mass parameter, the Curzon solution is

$$\mu = -\frac{m}{F}, \qquad \text{and} \qquad \nu = -\frac{mr^2}{2F^4}.$$

Looking at the metric, we see that there is no event horizon shielding the singularity at $r = 0, z = 0$. In fact, it is known that the singularity at $F = 0$ is directional, and the Curzon solution has an analytic continuation through $F = 0$. [16–18]

2.4.3 Finite line

From the Green's function given in the previous section, we can find harmonic functions with extended singularities. For instance, what, in Newtonian theory, would be a 'rod of constant density', gives us the solution

$$\mu = \frac{\rho}{2a}\int_{-a}^{a} \frac{\mathrm{d}z'}{\sqrt{(z-z')^2 + r^2}} = \frac{\rho}{2a}\log\frac{F_+ + F_- - 2a}{F_+ + F_- + 2a},$$

where $F_\pm = \sqrt{(z \pm a)^2 + r^2}$. This recovers the Curzon solution when $a \to 0$. Notice that there is now a singularity at $r = 0$ between $z = -a$ and $z = +a$. The associated conformal factor is

$$\nu = -\frac{\rho^2}{2a^2} \log \frac{4F_+F_-}{(F_+ + F_-)^2 - 4a^2}.$$

When $\rho = a$, Weyl observed that this solution is, in fact, the Schwarzschild metric. [14]

2.4.4 Two Curzon particles

Since Laplace's equation is linear, we may superimpose two Curzon solutions to obtain

$$\mu = -\frac{m_+}{F_+} - \frac{m_-}{F_-},$$

where $F_\pm = \sqrt{(z \pm a)^2 + r^2}$. To obtain the conformal factor, we have to integrate

$$\nu_z = 2r^2 \left[m_+^2 \frac{z+a}{F_+^6} + m_-^2 \frac{z-a}{F_-^6} + m_+ m_- \left(\frac{z+a}{F_+^3 F_-^2} + \frac{z-a}{F_+^2 F_-^3} \right) \right],$$

which gives

$$\nu = -\frac{r^2}{2} \left(\frac{m_+^2}{F_+^4} + \frac{m_-^2}{F_-^4} \right) + \frac{m_+ m_-}{2a^2} \frac{r^2 + z^2 - a^2 - F_+ F_-}{F_+ F_-}.$$

It may seem odd that we can present a 'static' solution that ostensibly contains two separate particles. However, notice that, as $r \to 0$,

$$\nu \to \frac{m_+ m_-}{2a^2} \frac{z^2 - a^2 - |z-a||z+a|}{|z-a||z+a|}.$$

This is zero when $|z| > a$ and non-zero when $|z| < a$. This has an important interpretation in terms of the metric, equation (5). The circles given by $r, z =$ constants have, for small r, a circumference of $2\pi r e^{-2\mu}$ and a radius of $r e^{2\nu - 2\mu}$. So, when $\nu(0, z)$ is nonzero, the metric has a conical singularity. Our computation shows that the line between the two particles has a conical singularity—and this could be interpreted as a stress holding the two particles apart from one another. [19]

3 Mass

A mass can be defined for asymptotically flat spacetimes in terms of the metric and its derivatives on a spacelike surface near infinity. Any quantity defined in this way is trivially conserved. However, the ADM definition of mass has the non-trivial property that it is positive for all spacetimes. In our treatment of static spacetimes, the ADM definition is not available since we have not specialised to asymptotically flat solutions. We make our own definition (similar to the Komar mass). We show that it is manifestly positive for static solutions and that it agrees with the Komar and ADM formulas in the appropriate cases. Our definition has the benefit of being manifestly positive and defined in arbitrary dimension.

3.1 Static mass

For static spacetimes we have, using the results of section 2.1,

$$0 = \int_M euR = \int_{\partial M} \hat{e} n(u),$$

where n is an outward pointing normal. Let Σ be the only component of ∂M which is not a horizon. We choose $u = 1$ on Σ. Let us define

$$Q = \int_{\Sigma} \hat{e} n(u) \qquad \text{or} \qquad Q = \int_{\Sigma} \star \mathrm{d}u.$$

It is a natural proposal for a mass, since it is always positive. This is because

$$Q = \int_{M} e \triangle u + \int_{H} \hat{e} n(u) = \int_{H} \hat{e} n(u),$$

and $n(u)$ is strictly positive when restricted to H by definition. (We take n to be the outward pointing normal on H.) So we have shown that $Q \geq 0$ with strict equality when H is empty. In sections 3.2 and 3.3 we will see that Q agrees with standard definitions of mass in the appropriate cases. The fact that u is the lapse function is largely irrelevant for the construction of Q. We could take any asymptotically constant harmonic function and define Q in the same way. This is similar to Witten's construction of the ADM mass using asymptotically constant spinors. [20] In fact, our construction is entirely homological. The pull back of $\omega = \star \mathrm{d}\psi$ to Σ is precisely $\hat{\star} n(\psi)$. So, Q is just the pairing

$$\langle [\Sigma], \omega \rangle ,$$

where $[\Sigma]$ is the class associated to Σ. It is interesting that the form ω should define the mass. It plays a prominent role in the multi-Taub-NUT instanton solutions, as mentioned in section 2.3. We now briefly compute Q in three examples.

3.1.1 Schwarzschild

Let us first work out Q for the Schwarzschild spacetime. We have

$$u = \frac{1 - \frac{a}{r}}{1 + \frac{a}{r}},$$

where $a = GM/2c^2$. For large r,

$$n(u) = \left(1 + \frac{a}{r}\right)^{-4} \frac{2a}{r^2} \simeq \frac{2a}{r^2}.$$

Then it follows that

$$Q = 8\pi a = \frac{4\pi G}{c^2} M.$$

In performing this calculation, one notices that Σ can be taken to be any sphere of constant radius r. Explicitly, we have

$$n(u) \mathrm{d}S^2 = 2a \, \mathrm{d}\Omega^2,$$

which is independent of the radius. This means that we need not make use of an asymptotically flat region at all, and can restrict our solution to a punctured 3-ball of any radius larger than $r = a$. We also have

$$Q = \lim_{r \to 0} \int_{\Sigma_r} \hat{e} n(u),$$

where Σ_r is the sphere around the origin of radius r.

3.1.2 Several black holes

For multiple horizons, we can choose the i^{th} horizon and define

$$Q_i = \int\limits_{H_i} \hat{e}n(u) = \lim_{r_i \to 0} \int\limits_{\Sigma_{r_i}} \hat{e}n(u).$$

For example, the multiple black hole initial data discussed in section 2.3 gives

$$Q_i = 8\pi a_i.$$

Then

$$Q = \sum Q_i,$$

which is the homological statement that $[\Sigma] = \sum[\Sigma_i]$.

3.1.3 Curzon

Choose the surface Σ to be the cylinder of radius $r = R$ and length $2R$ ($-R \leq z \leq R$) in Weyl's coordinates. Then

$$\int\limits_{\text{side}} \star \mathrm{d}u = \int\limits_{\text{side}} \mathrm{d}z\mathrm{d}\phi \frac{mR^2}{(z^2+R^2)^{3/2}} = 2\pi m\sqrt{2}$$

and

$$2\int\limits_{\text{top}} \star \mathrm{d}u = 2\int\limits_{\text{top}} \mathrm{d}r\mathrm{d}\phi \frac{mRr}{(r^2+R^2)^{3/2}} = 2\pi m(2-\sqrt{2}).$$

So, for this example,

$$Q = 4\pi m,$$

which justifies our calling m a 'mass parameter'.

3.2 Relation to Komar's mass

Let N be a stationary spacetime with a timelike Killing vector k. Associated to k is a 1-form θ given by the lowering the index with the metric. Then the 'Komar mass' is [21]

$$M_K = -\frac{1}{8\pi}\int_\Sigma \star_4 \mathrm{d}\theta,$$

for a closed 2-surface Σ. It is an intrinsically 4-dimensional definition. Much like our construction for static spacetimes, it is manifestly invariant under deformations of Σ and is thus homological. In our static case,

$$k = \frac{\partial}{\partial t},$$

and so $\theta = -u^2\mathrm{d}t$. Then

$$\mathrm{d}\theta = -2u\mathrm{d}u\mathrm{d}t.$$

The 4-volume is

$$e^{(4)} = u\mathrm{d}t \wedge e^{(3)}.$$

So we see that, for Σ a 2-surface in some $t = $ const. hypersurface,

$$M_K = \frac{1}{4\pi}\int_\Sigma \star_3 \mathrm{d}u.$$

That is,

$$M_K = \frac{1}{4\pi} Q.$$

In fact, this shows that our definition of mass is precisely Komar's definition restricted to static spacetimes and written in a notation adapted to the static case. Our definition has the merit of being defined in arbitrary dimensions without adaptation.

3.3 Relation to the Arnowitt-Deser-Misner mass

In the asymptotically flat case, choosing Euclidean coordinates, the Riemannian metric g_{ij} tends to δ_{ij} for large r. The ADM mass is then

$$M_{ADM} = \frac{1}{16\pi} \lim \int \left(\partial_j g_{ij} - \partial_i g_{jj}\right) n^i \mathrm{d}S^2.$$

In the case of a conformally flat metric, $g_{ij} = \psi^{4/(n-2)} \delta_{ij}$, we have

$$M_{ADM} = -\frac{1}{4\pi} \frac{n-1}{n-2} \lim \int n(\psi)\, \mathrm{d}S^2 = -\frac{1}{4\pi} \frac{n-1}{n-2} Q.$$

For the general case, M_{ADM} continues to be proportional to Q. To prove this, we can invoke the result that, for asymptotically flat spacetimes, $M_{ADM} = M_K$. [22]

4 Optics

We now come to the main subject of these notes. In contrast to the general situation, the null geodesics of a static spacetime can be studied using techniques from Riemannian geometry. This allows us, in section 4.2, to give an explicit method for computing image distortions on the sky of an observer in a static spacetime. Even without doing any numerical work, it is often possible to put bounds on the magnitude of the image distortions. In section **??**, we review another technique from Riemannian geometry—the Gauss-Bonnet theorem. This can be applied to computing deflection angles, as recently pointed out by Gibbons and Werner. We also point out that this could be used to study optics in axisymmetric spacetimes and do some computations.

4.1 The space of static null rays

For a Lorentzian metric on $M \times S^1$,

$$\hat{g} = -u^2 \mathrm{d}t^2 + g,$$

Fermat's principle states that null geodesics minimize the functional

$$S = \int \mathrm{d}t = \int \sqrt{u^{-2} g}.$$

Equivalently, this means that the projection of a null geodesic onto M is a geodesic of the metric

$$g_o = u^{-2} g,$$

which is called the optical metric. Let ∇^o be the associated Levi-Civita connection. A basic observation is that geodesics satisfy what we call the 'reciprocity relation'. (The Lorentzian relation first appeared in [23].) Given a g_o-geodesic with momentum k, let l_1, l_2 be Jacobi fields. That is,

$$\nabla^o_k l - \nabla^0_l k = 0.$$

It is easy to see that the Wronskian

$$Wr(l_1, l_2) = l_1 \cdot \nabla^o_k l_2 - l_2 \cdot \nabla^0_k l_1$$

is a constant along the geodesic. This follows from Jacobi's equation. Now, choose a frame e_1, e_2 of vector fields on the geodesic satisfying $e_i \cdot e_j = \delta_{ij}$ and $e_i \cdot k = 0$. We parallel transport e_i up the geodesic. Then any pair of Jacobi fields can be represented by a square matrix of coefficients as

$$\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = M \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

Taking a derivative with ∇^o_k we see that

$$\begin{bmatrix} \dot{l}_1 \\ \dot{l}_2 \end{bmatrix} = \dot{M} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

A pair of Jacobi fields represents a two dimensional bundle of adjacent geodesics. At a point, $\det M$ describes the focussing of the bundle (i.e. $\det M = 0$ means that the point is a focal point of the bundle) and $\det \dot{M}$ describes the size of the bundle's image on the sky of the point (i.e. $\det \dot{M} = 0$ would happen if all the rays in the family were collinear). Then consider any two families, M_1 and M_2, of adjacent geodesics. We may define the extended Wronskian

$$Wr(M_1, M_2) = M_1^T \dot{M}_2 - M_2^T \dot{M}_1,$$

and observe that it is constant along the geodesic. Then, if p is a point at which $\det M_1 = 0$ and q is a point at which $\det M_2 = 0$, we have the following relation

$$\frac{\det M_1(q)}{\det \dot{M}_1(p)} = \frac{\det M_2(p)}{\det \dot{M}_2(q)}.$$

The quantity appearing on the left hand side is the ratio of the size of an illuminated object at q to the size of the image of that object at p. The quantity on the right hand side is the converse: the ratio of the size of an object at p to the size of its image at q. That these two quantities are equal to each other is, I believe, an important principle in astronomy.

4.2 Image distortion

We now want to solve the following problem. Suppose that we have a static spacetime $M \times \mathbb{R}$ with an optical metric h on M.

Problem: determine the image distortion on the sky of some observer in this spacetime.

As stated, this problem is not very precise. However, in solving the problem, we will also formulate it in a precise way. Fix a point $p \in M$ to be the observer. Our idea is to use geodesic coordinates for M based at p. Geodesic coordinates were introduced in the prologue, section 1. Suppose we introduce spherical coordinates on T_pM which we call s, θ^i, where s is the radius and θ^i are angles. The flat metric on T_pM is

$$\mathrm{d}s^2 + s^2\Omega,$$

where Ω is the constant curvature metric on the $n-1$-sphere. As discussed in the prologue, the exponential map allows us to use these as coordinates on the neighbourhood of p in M. The surfaces s = constant are the surfaces of constant geodesic distance from p. The normal vector field to the surfaces is

$$N = \frac{\partial}{\partial s}.$$

The induced metric on the surfaces is $\gamma_{ab} = h_{ab} - N_a N_b$. Near p, the metric γ is the spherical metric. For some finite distance $s = s_1$, γ will no longer be the spherical metric. The difference in γ alone some line of sight is the distortion between the emitting object positions and the received image positions.

Refined problem: determine the Jacobian, $\det \gamma(s_1)/\det \gamma_o$, between the flat metric γ_o of the sphere in T_pM and the induced metric γ on the sphere $s = s_1$ in M. All quantities are evaluated on some line of sight given by constant θ^i.

4.2.1 Riccatti's equation

We solve the problem by studying the extrinsic curvature of the surfaces s = constant. This is given by

$$K_{ab} = \nabla_a N_b,$$

or, equivalently, by[7]

$$K_{ab} = \frac{1}{2}\frac{\partial}{\partial s}\gamma_{ab}.$$

Taking contractions shows that

$$\frac{\partial}{\partial s}\log\sqrt{\det\gamma} = K.$$

We compare the distorted metric γ to the metric on the spheres s = constant in T_pM, which is $\gamma_o = s^2\Omega$. Then $\det\gamma_o = s^{2(n-1)}\det\Omega$ and

$$\frac{\partial}{\partial s}\log\sqrt{\det\gamma_o(s)} = \frac{n-1}{s}.$$

For any fixed direction in the sky, θ^i, we are interested in the Jacobian

$$J(s) = \frac{\sqrt{\det\gamma(s)}}{\sqrt{\det\gamma_o(s)}}.$$

For small s, $J(s) \to 1$. On the other hand,

$$\frac{\partial}{\partial s}\log J(s) = -\frac{n-1}{s} + K.$$

Integrating this equation we find that

$$J(s) = \frac{1}{s^{n-1}}\exp\left(\int_0^s \mathrm{d}s' K(s')\right). \tag{8}$$

This is our solution to the problem. To determine $K(s)$, one must solve the following equation, which is derived in appendix A,

$$K'(s) = -K_{ab}K_{ab} - R_{NN},$$

[7] To go between the two expressions we do the following,

$$(\nabla_{\partial_a}N)\cdot\partial_b = -N\cdot\nabla_{\partial_a}\partial_b = -\Gamma^s_{ab} = \frac{1}{2}\frac{\partial}{\partial s}h_{ab} = \frac{1}{2}\frac{\partial}{\partial s}\gamma_{ab}.$$

The first step is the fact that N and $\partial/\partial\theta^a$ are everywhere orthogonal. The last step follows by the geodesic equation $\nabla_N N_a = 0$.

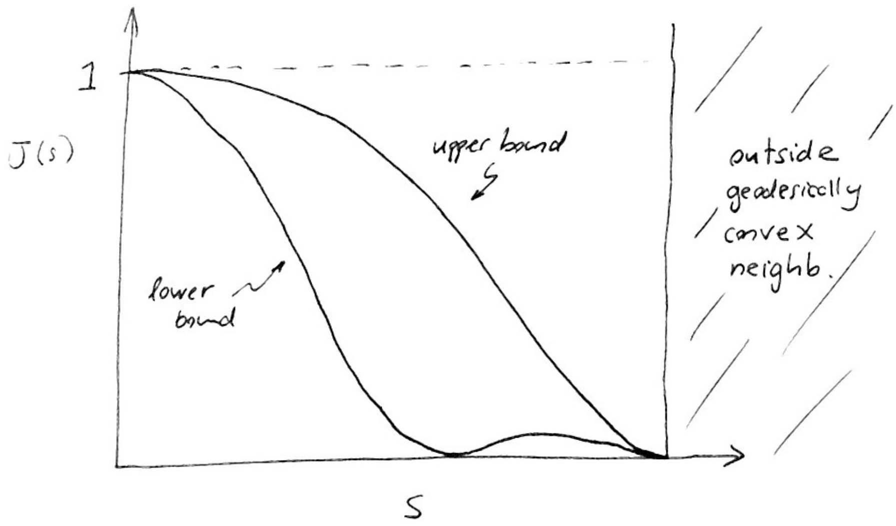

Figure 2: Upper and lower bounds on the Jacobian given lower and upper (positive) bounds on the curvature.

where R_{NN} is the Ricci tensor contracted with N^aN^b. As it stands, it may seem intractable to integrate this equation. However, we can convert this equation into two ordinary differential equations which are easily integrable. To do this, we begin with

$$\frac{\partial}{\partial s}K^a{}_b = -K^a{}_cK^c{}_b - R_N{}^a{}_{bN}.$$

This result implies the previous equation upon contraction. We can regard $K^a{}_b$ as a linear operator acting on vectors which are tangent to the surfaces (orthogonal to N). Let its eigenvectors be $e_i(s)$ with eigenvalues $\lambda_i(s)$. Then the eigenvalues evolve according to the equations

$$\dot{\lambda}_i(s) + \lambda_i^2 = -\kappa_i(s),$$

where $\kappa_i = R_{Ne_ie_iN}$. By solving these equation we can find $K(s)$ as the sum of eigenvalues,

$$K(s) = \sum_i \lambda_i(s).$$

We use this formulation to form estimates of $J(s)$ in the following subsection.

4.2.2 Estimates of the image distortion

If it is possible to place bounds on κ_i, then we can place bounds on the Jacobian $J(s)$. Suppose, for instance, that we have a bound

$$0 < a^2 < \kappa_i,$$

for some constant a. Then we have the inequality

$$\dot{\lambda} < -\lambda^2 - a^2,$$

which integrates to give

$$\int_{-\infty}^{\lambda} \frac{\mathrm{d}\lambda}{\lambda^2 + a^2} < -s.$$

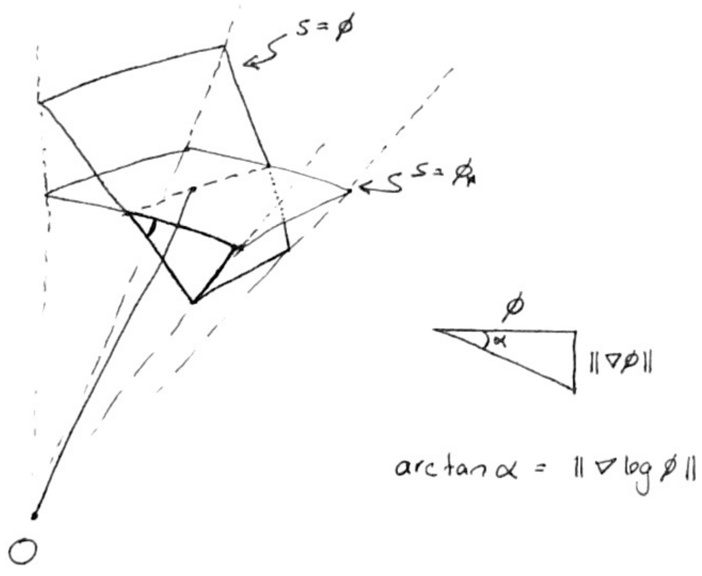

Figure 3: Computing the relationship between Ω and $\tilde{\gamma}_o$.

We require the boundary condition $\lambda \to -\infty$ as $s \to 0$ since $K(s)$ diverges as $s \to 0$. The integration gives

$$\lambda < a \tan\left(-as - \frac{\pi}{2}\right) = a \cot(as).$$

If the static spacetime has $n+1$ dimensions, then we have the bound

$$K < (n-1)a \cot(as).$$

Integrating our result, equation (8), we find

$$J(s) < \left(\frac{\sin(as)}{as}\right)^{n-1}.$$

Notice that $J \to 0$ before $s = \pi/a$. So, by putting a lower bound on the curvatures κ_i we have found an upper bound on the region of geodesic convexity around p. Likewise, an upper bound $\kappa_i < b^2$ gives a lower bound

$$J(s) > \left(\frac{\sin(bs)}{bs}\right)^{n-1},$$

which loses its power for distances s larger than π/b.

4.2.3 Distorted emission surfaces

In applications, it is unlikely that the objects emitting light are all at constant geodesic distance from the observer at p. Fortunately, our foregoing analysis can be easy adapted to this situation. Suppose that the objects emitting light are located at the sphere $s = \phi$, where ϕ is some function of the angles θ^i. This emission surface is reached by the same geodesics as before (i.e. those tangent to N), but with a new flow equation

$$\frac{\partial}{\partial t} x^a = \phi N^a,$$

where the new parameter t satisfies

$$\phi = \frac{\partial s}{\partial t}.$$

Let $\tilde{K}$ be the extrinsic curvature of the surfaces $s = t\phi$, and let $\tilde{\gamma}$ be the metric on these surfaces. Then the associated ("rate altered") Riccatti equation is

$$\frac{\partial}{\partial t}\tilde{K}(t) = -\triangle_\Omega \phi - \phi\left(\tilde{K}_{ab}\tilde{K}_{ab} + R_{NN}\right).$$

The laplacian term, $\triangle_\Omega \phi$, is the laplacian of ϕ regarded as a function on the unit sphere with constant curvature metric Ω. The evolution of $\log\sqrt{\det\tilde{\gamma}}$ is related to $\tilde{K}$ just as before. Let $\tilde{\gamma}_o$ be the metric on the surfaces $s = t\phi$ in T_pM. Then

$$\frac{\partial}{\partial t}\log\sqrt{\frac{\det\tilde{\gamma}(t)}{\det\tilde{\gamma}_o(t)}} = -\frac{n-1}{t} + \tilde{K}.$$

However, we must also determine the relationship between $\gamma_o(t)$ and the spheres $s = \text{constant}$. Heuristically, we can do this in two steps. First, fix a direction θ^i_* at which ϕ takes the value ϕ_*. Then compare the surface $s = \phi$ to the sphere $s = \phi_*$ at θ^i_*. The two planes tangent to the surfaces make an angle

$$\alpha = \arctan\left(||\nabla \log\phi||\right).$$

We infer that (see the figure)

$$\det\tilde{\gamma} = t^2\phi^2\sqrt{1 + ||\nabla\log\phi||^2}\,\det\Omega.$$

So the total image distortion is given by solving

$$\frac{\partial}{\partial t}\log J(t) = -\frac{n-1}{t} + \tilde{K},$$

with the boundary condition

$$J(t) \to (n-1)\log\phi + \frac{1}{2}\log\left(1 + ||\nabla\log\phi||^2\right),$$

as $t \to 0$. For example, consider only a small distortion $\phi = 1 + f$, for some small function f. Then, to first order, the Jacobian has the boundary behaviour

$$J(t) \to 1 + (n-1)f + \mathcal{O}(f^2).$$

In effect, the new problem is the same as the old one, but with a different curvature tensor, namely we replace

$$R_{NabN} \mapsto R_{NabN} + \nabla_a\nabla_b\phi.$$

The methods we used to bound $J(t)$ in the previous subsection can thus also be applied to the present case, with this replacement.

4.3 Optics in the equatorial plane

In the previous section, we discussed image distortion by means of differential geometric considerations. We now briefly mention a different approach. It was recently pointed out by Gibbons and Werner [24] that one can infer several properties of the ray geometry—such as deflection angles—from topological considerations via the Gauss-Bonnet theorem. The idea is very simple. Consider a surface with Gauss curvature R. The Euler characteristic of a region D of the surface, possibly with corners, is given by

$$\pi\chi(D) = \int_D R + \int_{\partial D} \kappa + \sum_i \alpha_i,$$

where κ is the geodesic curvature of the boundary[8] and α_i are the angles of the corners. It is possible to use this to find the deflection angles of geodesics by considering regions D which are bounded by geodesics (for which $\kappa = 0$). The metric of a surface can always, at least locally, be made conformally flat,

$$g = \varphi^2(\mathrm{d}x^2 + \mathrm{d}y^2).$$

Then the Gauss curvature of the surface is given by Liouville's equation

$$R = -\frac{1}{\varphi^2}\partial^2 \log\varphi.$$

This makes the computations very straightforward. We will illustrate the idea with two examples.

4.3.1 Schwarzschild

For Schwarzschild the optical metric is, recalling section 2.3.1,

$$g_o = \frac{\left(1+\frac{a}{r}\right)^4}{\left(1-\frac{a}{r}\right)^2}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right).$$

By spherical symmetry, we can consider null geodesics restricted to the equatorial plane. The Gauss curvature of the plane is

$$R\sqrt{|g_o|} = -\frac{a}{r^2(1-a/r)^{3/2}}\left(1-\frac{3a}{4r}\right) \simeq -\frac{a}{r^2} + \mathcal{O}(r^{-3}).$$

We can then use Gauss-Bonnet to estimate the deflection angle for a null geodesic at large impact parameter. Approximating the geodesic by the straight line $(r,\phi) = (b/\cos\phi, \phi)$, we consider the region D outside this line, bounded by a large circle near $r = \infty$. Clearly $\chi(D) = 2$. Let α be the two angles made by the geodesic when it intercepts the circle near infinity. Then

$$2\pi = 2\alpha - \int_{-\pi/2}^{\pi/2} \mathrm{d}\phi \int_{b/\cos\phi}^{\infty} \mathrm{d}r \frac{a}{r^2},$$

and

$$\pi - \alpha = -\frac{2a}{b} = -\frac{4m}{b},$$

which is the usual answer.

4.3.2 Axisymmetric spacetimes

Consider now a general axisymmetric static spacetime in Weyl form. Since $\partial/\partial\phi$ is a Killing vector, we can restrict ourselves to studying null geodesics which begin and remain in the meridional plane $\phi = 0$. The induced metric on this plane is, in the notation of section 2.4,

$$g = e^{2\nu-2\mu}(\mathrm{d}z^2 + \mathrm{d}r^2),$$

[8] If v is tangent to ∂D, then $\kappa = \nabla_v v$, where ∇ is the Levi-Civita connection for the metric on the surface.

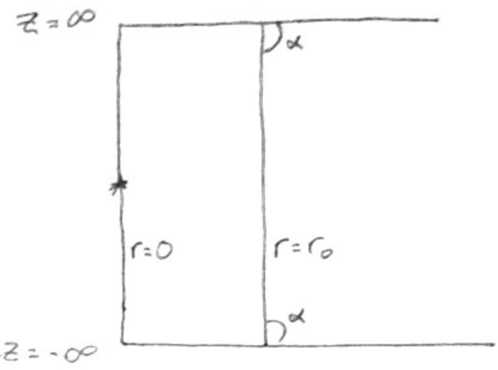

Figure 4: Finding angle deficits in the meridional plane of the Curzon metric. The mass is located at the asterisk. We have compactified in the z direction, but not in the r direction.

and the optical metric is

$$g_o = e^{2\nu-4\mu}(\mathrm{d}z^2 + \mathrm{d}r^2).$$

We now determine the Gauss curvature R of g_o. The equations of motion described in section 2.4 lead us to the following simple formula,[9]

$$R = -e^{4\mu-2\nu}\left(\frac{2}{r}\mu_r - (\mu_r^2 + \mu_z^2)\right).$$

For example, the Curzon solution gives

$$R = -e^{4\mu-2\nu}\left(\frac{2m}{F^3} - \frac{m^2}{F^4}\right).$$

For large F, the leading contribution is

$$R \simeq -\frac{2m}{F^3} + ...$$

Approximating a geodesic by the straight line $r = r_o$, we compute that

$$-\int_{-\infty}^{\infty} \mathrm{d}z \int_{r_o}^{\infty} \mathrm{d}r \frac{2m}{(z^2 + r^2)^{3/2}} \simeq -\frac{2m}{r_o}.$$

By the same argument as earlier, we find that the null geodesic approaches infinity with an angle deficit of

$$\pi - \alpha = -\frac{m}{r_o},$$

which is similar to the case we studied for Schwarzschild.

[9]The intermediate step is to show that

$$\nu_{zz} = 2r\mu_{zz}\mu_r + 2r\mu_z\mu_{rz} \qquad \text{and} \qquad \nu_{rr} = 2r\mu_r\mu_{rz} - 2r\mu_z\mu_{rs} + \mu_r^2 - \mu_z^2.$$

A Riccatti's equation

The purpose of this section is to present a slick derivation of Riccatti's equation. For a detailed discussion of this and related matters, see [10]. Riccatti's equation describes the evolution of the extrinsic curvature of a family of surfaces arising from geodesic flow.[10] The key idea of the derivation is to express the extrinsic curvature as the laplacian of a scalar. To begin, consider some family of surfaces with normal vector field $N = \partial/\partial s$. That is, the surfaces are given by $s =$ constant. By assumption, N_a is geodesic. The induced metric on the surfaces is

$$h_{ab} = g_{ab} - N_a N_b,$$

and the Laplacian of g can be decomposed as

$$\triangle_g = h^{ab}\nabla_a\nabla_b + \nabla_N\nabla_N - N^a(\nabla_a N^b)\nabla_b.$$

On the other hand, the laplacian of h is

$$\triangle_h = h^{ab}D_a D_b = h^{ab}\nabla_a\nabla_b - h^{ab}\nabla_a(N_b N^c)\nabla_c,$$

where $D_a = h_a^b\nabla_b$ are projected covariant derivatives. Recall that N_a is geodesic. Then we find that the laplacian decomposes in the following way,[11]

$$\triangle_g = \triangle_h + \nabla_N\nabla_N + K\nabla_N,$$

where $K = \nabla_a N^a$ is the trace of the extrinsic curvature of the surfaces. Notice, then, that

$$\nabla_g s = K,$$

where s is the scalar defining the foliation. For any scalar field f the following identity holds,[12]

$$\nabla_a\triangle f = \triangle\nabla_a f - R_{ac}\nabla_c f.$$

Using this identify for the scalar s, we immediately find that

$$\nabla_N K = N_a\triangle N_a - R_{NN}.$$

Equivalently,

$$\nabla_N K = -K_{ab}K_{ab} - R_{NN},$$

where $K_{ab}K_{ab} = \nabla_a N_b\nabla_a N_b$. This is Ricatti's equation. A slightly longer calculation gives the following result,

$$\nabla_N K_{ab} = -K_{ac}K_{bc} - R_{NabN},$$

which implies the contracted version derived above.

[10]This is not the problem Riccatti studied. Nevertheless, the equation here bears a closer relationship with Riccatti's ODE, $f'(x) + f(x)^2 = g(x)$.

[11]The intermediate step, not assuming that N is geodesic, is given by

$$\triangle_g = \triangle_h + \nabla_N\nabla_N + \nabla_a(N^a N^b)\nabla_b - \nabla_N N^a\nabla_a - N^b\nabla_N(N_b N^c)\nabla_c.$$

[12]To prove the identity, notice that

$$\nabla_a\nabla_b\nabla_b f = \nabla_b\nabla_a\nabla_b f - R_{babc}\nabla_c f = \nabla_b\nabla_b\nabla_a f - R_{ac}\nabla_c f,$$

by the definition of the curvature tensor.

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