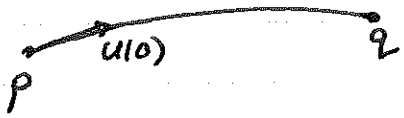
Light Cones

Fix p in M. We can take any vector $u(0)$ in T_pM as an initial velocity. Integrating the geod. equation for unit time gives a point q in M proper distance $|u(0)|$ from p.



So, the radius function $s = |u(0)|$ on T_pM defines a scalar on M near p.

Moreover, we have spherical polars on T_pM: s, θ^A so that $\frac{\partial}{\partial s}, \frac{\partial}{\partial \theta^A}$ are basis for TT_pM: here, $\frac{\partial}{\partial s} \cdot \frac{\partial}{\partial \theta^A} = 0$ in Euclidean metric

$$ds^2 + \gamma_{AB}\, d\theta^A d\theta^B$$

on T_pM. Varying θ^A gives a family of geodesics and since

$$\frac{\partial}{\partial s}\left(\frac{\partial}{\partial s} \cdot \frac{\partial}{\partial \theta^A}\right) = (\pounds_{\partial_s} \partial_s) \cdot \partial_A + \partial_s \cdot (\pounds_{\partial_s} \partial_A) = 0$$

$$\text{or} \quad (\nabla_{\partial_s} \partial_s) \cdot \partial_A + \partial_s \cdot (\nabla_{\partial_s} \partial_A) = 0$$

we see that ∂_A remains orthogonal to ∂_s.

Area Jacobian. Let N be the normal to the geodesic spheres. Then, acting on tangent vectors ∂_A,

$$K_{AB} = K(\partial_A, \partial_B) \equiv \nabla_A N_B = (\nabla_{\partial_A} N) \cdot \partial_B$$

$$= - N \cdot \nabla_{\partial_A} \partial_B$$

$$= - \Gamma^s{}_{AB}$$

$$= \frac{1}{2} \frac{\partial}{\partial s} g_{AB}$$

$$= \frac{1}{2} \frac{\partial}{\partial s} h_{AB}$$

$$h_{ab} = g_{ab} - N_a N_b$$
$$h_{AB} = g_{AB} - \underset{0}{\underset{\shortparallel}{N_A N_B}} .$$

and

$$g^{ab} K_{ab} = \nabla_a N^a$$

$$= \frac{1}{2} g^{ab} \partial_s h_{ab}$$

$$= \frac{1}{2} \partial_s \log \det h$$

$$\partial_s \log \sqrt{\det h} = K .$$ — nb. singular small s.

For the spheres in $T_p M$, $\sqrt{\det h_0} = s^2 \sin\theta$. So

$$\partial_s \log \sqrt{\det h_0} = \frac{1}{s} .$$

We compare $|h| / |h_0| \to 1$ or $r \to 0$. And

$$\partial_s \log \sqrt{|h|} / \sqrt{|h_0|} = - \frac{1}{s} + K .$$

Call the ratio J, (for Jacobian!)

$$\log J(s) - \log \underset{\underset{1}{\shortparallel}}{J(0)} = - \log s + \int^s K \Big|_0^s$$

$$J(s) = \frac{1}{s} \exp\left(\int_0^s ds\, K \right).$$

Near $s = 0$, $\quad -\log s + \int K = -\log s + \log s = 0 .$

<u>Laplacians</u>. We can expand

$$\Delta = g^{ab}\nabla_a\nabla_b \qquad\qquad (h_{ab} = g_{ab} - N_a N_b)$$

$$= h^{ab}\nabla_a\nabla_b + N^a N^b \nabla_a\nabla_b$$

$$= h^{ab}\nabla_a\nabla_b + \nabla_N\nabla_N - N^a(\nabla_a N^b)\nabla_b$$

Also

$$\Delta_h = h^{ab} D_a D_b \qquad\qquad D_a = h_a{}^c\nabla_c$$

$$= h^{ab} h_a{}^c \nabla_c (h_b{}^d \nabla_d \,\cdot\,)$$

$$= h^{ab}\nabla_a\nabla_b \,\cdot\, - h^{ab}\nabla_a(N_b N^c)\nabla_c$$

since

$$h^{ab} h_a{}^c = (g^{ab} - N^a N^b) h_a{}^c = h^{cb}$$

then

$$\Delta = \Delta_h + \nabla_N\nabla_N + \nabla_a(N^a N^b)\nabla_b - N^a(\nabla_a N^b)\nabla_b$$

$$- N^b\nabla_N(N_b N^c)\nabla_c \,.$$

Since N is geodesic

$$\Delta = \Delta_h + \frac{\partial^2}{\partial s^2} + K\frac{\partial}{\partial s}$$

and, in particular,

$$\Delta s = K \,.$$

Ricatti's Equation. We then easily see

$$\nabla_a \Delta S = \nabla_a \nabla_b \nabla_b S$$

$$= \nabla_b \nabla_a \nabla_b S - R_{babc} \nabla_c S$$

$$= \Delta \nabla_a S - R_{ac} \nabla_c S .$$

This holds for any scalar. But for S we have

$$\nabla_N \Delta S = N_a \Delta N_a - R_{NN}$$

$$= N_a \nabla_b \nabla_b N_a$$

$$\parallel$$

$$\nabla_b (N_a \nabla_b N_a) - \nabla_b N_a \nabla_b N_a .$$

So

$$\nabla_N K = - K_{ab} K^{ab} - R_{NN} .$$

An elementary (less pretty) calculation gives

$$\nabla_N K_{ab} = - K_{ac} K^c{}_b - R_{NabN} .$$

A Bound on the Jacobian. Suppose e is a normalised eigenvector of $K^a{}_b$, orthogonal to N. Then its eigenvalue evolves as

$$\dot{\lambda} = - \lambda^2 - \kappa \qquad \kappa = R_{NeeN} .$$

If we can bound $\kappa > a^2 > 0$ then we have

$$\dot{\lambda} < - \lambda^2 - a^2$$

$$\int_{-\infty}^{\lambda} \frac{d\lambda}{\lambda^2 + a^2} < -s \quad \ldots \quad \text{note boundary condition } \lambda \to -\infty \text{ as } s \to 0.$$

$$\Rightarrow \quad \frac{1}{a}\tan^{-1}\left(\frac{\lambda}{a}\right) + \frac{1}{a}\frac{\pi}{2} < -s$$

$$\Rightarrow \quad \lambda < a\tan(-as - \pi/2)$$

$$= a\cot(as).$$

Then K is sum of eigs so

$$\partial_s \log J = -2s^{-1} + K$$

$$\leq -2s^{-1} + 2a\cot(as)$$

$$\log J \leq -2\log s + 2\log\sin(as)$$

$$J \leq \left(\frac{\sin(as)}{s}\right)^2.$$

Note that $J \to 0$ before $s \to \pi/a$. So lower bound on principal curvatures also bounds the region of geodesic convexity.

<u>Rate Altered Ricatti</u>. The emission surface is unlikely to be a sphere of constant s. Suppose it is located at $s = \phi$, for ϕ a function of the angles θ^A. It is reached by the same geodesics (tangent to N), but with a new flow equation,

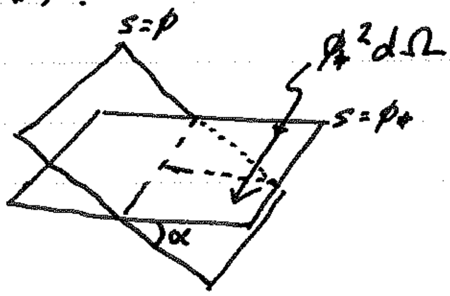
$$\frac{\partial x}{\partial t} = \phi N \qquad \left(\phi = \frac{\partial s}{\partial t}\right)$$

The associated Ricatti equation is

$$\partial_t \breve{K} = - \Delta_\sigma \phi - \phi (\breve{K} \breve{K} + R_{NN})$$

The evolution of $\log \sqrt{\det h}$ is as before. However, we are less certain of the boundary condition: $\breve{K}$ at small t.

<u>Supertranslations</u>. Inside $T_p M$ consider the deformed spheres $s = \phi$, for ϕ an angular function. The area element can be worked out in two steps. Fix a direction, θ^A_*, at which $\phi = \phi_*$. Then compare $s = \phi$ to the sphere $s = \phi_*$ at the point (ϕ_*, θ^A_*):



the angle α is

$$\tan \alpha = \|\nabla \phi\| / \phi$$

so the hypotenuse is $\sqrt{\phi^2 + \|\nabla\phi\|^2}$ and

$$d\breve{\Omega} = \phi \cdot \text{hyp}\, d\Omega = \phi^2 \sqrt{1 + \|\nabla \log \phi\|^2}\, d\Omega .$$

Considering the family $s = t\phi$,

$$d\breve{\Omega} = t^2 \phi \sqrt{\phi^2 + \|\nabla \phi\|^2}\, d\Omega$$

which has a uniform ratio with the spheres $t^2 d\Omega$.

This suggests

$$\partial_t \log \frac{\sqrt{|h|}}{d\Omega} = -t^{-1} + \overset{n}{K}$$

with boundary condition

$$\log \frac{\sqrt{|h|}}{d\Omega} \to 2\log\phi + \frac{1}{2}\log\left(1+|\nabla\log\phi|^2\right)$$

as $t \to 0$.

Consider only a small supertranslation

$$\phi = 1 + f$$

to leading order

$$\dot{K} = -\Delta f \cancel{+} - (KK + \tilde{R})$$

$$\phi^2\sqrt{1+|\nabla\log\phi|^2} = (1+2f+\cdots)\left(1+\frac{1}{2}|\nabla f|^2+\cdots\right)$$

$$= 1 + 2f + \frac{1}{2}|\nabla f|^2$$

For the evolution of cuts at $\mathcal{J}^+$ in asym flat spaces, Strominger & Zhiboedov (and others in the 70s...) observed that the metric change induced by 1st order curvature can be reproduced as the change induced by a supertranslation. We do not know if a similar result holds here (in the static case).

Reciprocity Relation

A tangent vector ~~the~~ to $\mathbb{A}$ is a Jacobi field based at some ray $\Gamma \in \mathbb{A}$. We can describe the symplectic structure on $\mathbb{A}$ in terms of Jacobi fields,

$$\omega(J_1, J_2) = J_1 \cdot \dot{J}_2 - \dot{J}_1 \cdot J_2$$

which does not depend on where it is evaluated on Π since

$$\nabla_u \omega(J_1, J_2) = J_1 \cdot \ddot{J}_2 - \ddot{J}_1 \cdot J_2$$

$$= -\langle J_1, R(u, J_2)u \rangle + \langle J_2, R(u, J_1)u \rangle$$

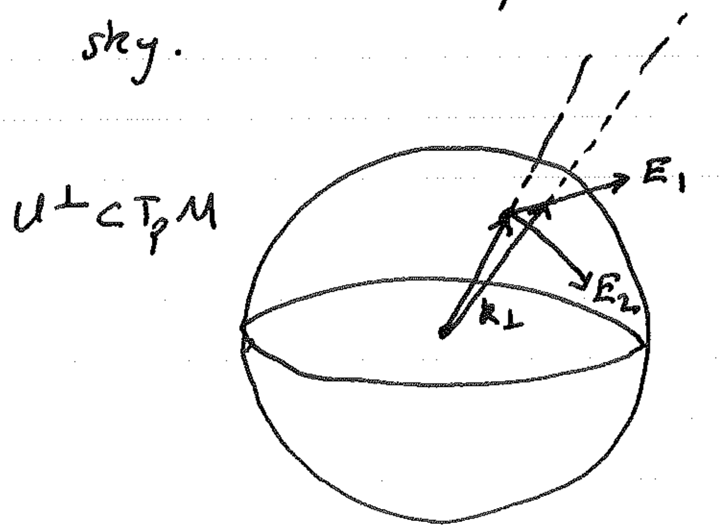
$$= 0 .$$

A Jacobi field is specified by $J(0)$ and $\dot{J}(0)$ — Jacobi's equation being second order. Moreover, the null space of K is 3 dimensional. But quotienting out by $V \sim V + K$ shows that the nontrivial Jacobi fields have $J(0), \dot{J}(0)$ in a 2 dim vector space. This can be represented as ~~the~~ a subspace of T_pM by choosing an arbitrary timelike vector u. Then choose E_1, E_2 (orthogonal, normalised) such that $E_A \cdot u = 0$. We transport this along Π so that

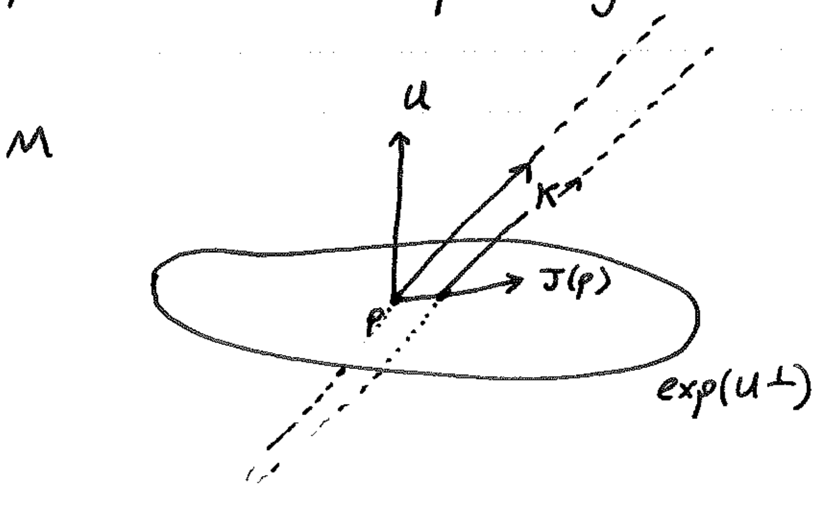
$$\nabla_K (E \cdot u) = 0 \quad \Leftrightarrow \quad \nabla_K E = -K \left(\frac{E \cdot \nabla_K u}{K \cdot u} \right).$$

Since $E_A \cdot K = 0$, $E_A \cdot \nabla_K E_B = 0$ and the frame remains orthogonal.

Choosing two Jacobi fields, we can regard their span as describing a bundle of rays near Π. In E_A components, write this as a matrix M. As discussed in sec. 1, the sky of the observer u is the unit sphere in ~~the~~ $u^\perp \subset T_pM$. E_A can then be regarded as target vectors on the sky, and the 2 dim family of Jacobi fields with $J(p) = 0$ correspond to other rays in the null cone of p with displaced position on p's sky.



$J(p) \neq 0$ describes a neighbouring ray whose intersection with the orthogonal neighbourhood of U in M is displaced from p by $J(p)$.



<u>Reciprocity</u>. In matrix form the symplectic form is

$$\omega = M_1(s)^T \dot{M}_2(s) - M_2^T(s)\dot{M}_1(s).$$

We fix an <u>E</u>mitter and <u>R</u>eciever. Suppose we set $M_1(E) = 0$, so M_1 describes the null cone bundle through E around T'. Then, an angular displacement $\theta^A E_A$ corresponds to choosing the linear combination

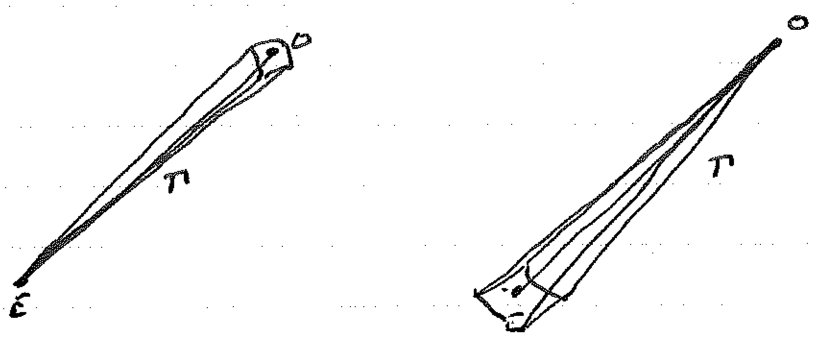
$$\dot{M}_1(E)^{-1}\theta \qquad (\text{choosing } |k_\perp| = 1).$$

of the two Jacobi fields. At O, this combination spans a displacement

$$\underline{\ell} = M_1(O)\,\dot{M}_1(E)^{-1}\,\underline{\theta}.$$

The area Jacobian is

$$\det J_{EO} = \det M_1(O)\,\dot{M}_1(E)^{-1}.$$



$D_{lum} = \det J_{EO}$ $\qquad$ $D_{ang} = \det J_{OE}$.

We chose $|k_\perp| = 1$ which, using $U, \hat{k}_\perp, E_1, E_2$ as a tetrad, shows that $|K\cdot U| = 1$. If we had allowed $|K\cdot U|$ arbitrary, the ~~angular~~ angle subtended by $\theta^A E_A$ at Emitter would be changed by a factor of $1/|K\cdot U|_E$. So, allowing for K to have arbitrary scale we ~~are~~ have

$$D_{lum} = \det J_{EO}\,|K\cdot U|_E$$

$$D_{ang} = \det J_{OE}\,|K\cdot U|_O$$

Finally, evaluating ω at E and O gives

$$M_1(O)^T\dot{M}_2(O) = -M_2^T(E)\,\dot{M}_1(E)$$

$$\Rightarrow \quad \det J_{EO} = \det J_{OE}.$$

Shear. Notice now that an angular separation $\theta^A E_A$ at E gives a physical separation ℓ^A of length

$$\ell^A \ell_A = J^A{}_B J_{AC} \theta^B \theta^C .$$

If ~~$\hat{\theta}$~~ $\theta^A \theta_A = \theta^2$, we get

$$\ell / \theta = \sqrt{J^A{}_B J_{AC} \hat{\theta}^B \hat{\theta}^C}$$

which is direction dependent. We have two cases

(i) $\lambda_1 = \lambda_2$: $\ell/\theta = \det J_{OE} = D_{ang}$

(ii) $\lambda_1 \neq \lambda_2$: $\ell/\theta \neq D_{ang}$ for most $\hat{\theta}$.

Here, λ_1, λ_2 are the eigs of J_{OE}. Since light cone congruences generically have shear, D_{ang} is not really a 'angular-diameter distance'.

Static Reciprocity. Observe that the proper distance spheres of the previous section are compatible with reciprocity. $R_{NN} = R_{\tilde{N}\tilde{N}}$, where $\tilde{N}$ is the reversed orientation tangent vector. Near E and O we give K_{ab} the same initial condition. Since $K = \lambda_1 + \lambda_2$, we reduce to ~~#~~ solving for λ_i and $\tilde{\lambda}_i$. While we do not expect $K(s) = \tilde{K}(1-s)$, ~~we see~~ we do have

$$\lambda(1) - \lambda(0) = \int_0^1 \lambda^2 + R_{NN}$$

$$\int_0^1 R_{NN} = - \int_1^0 R_{\tilde{N}\tilde{N}} \qquad \text{etc.}$$

Conjecture. It follows that $|\int K| = |\int \tilde{K}|$.

<u>Volumes</u>. Using our (partly) normalised orthogonal frame $U, \hat{K}^{\perp}, E^1, E^2$ we see that the "volume" of the intersection of a pencil around T' is

$$e^{(3)} = \hat{K}^{\perp} \wedge E^1 \wedge E^2, \qquad \mathrm{Vol} = |U \cdot K| \det(M) \, e^{(3)}.$$

Can we find the ratio $\det M(0) / \det M(E)$ for an arbitrary pencil? Using only the symplectic invariant I would find $\frac{1}{2}n(n-1)$ matrix equations if I considered M together with $n-1$ other pencils. On the other hand, I would have $2n$ unknown matrices: if I specified each $M, \dot{M}$ at E, I would hope to determine each $M, \dot{M}$ at 0. Now,

$2n$	$\frac{1}{2}n(n-1)$
4	1
6	3
8	6
10	10

So generically I must consider 5 ~~[illegible]~~ pencils. How many linearly independent pencils are there? For a given M, ~~we~~ can multiply $M \mapsto MP$ for nonsingular P to obtain the same pencil. For $M \neq 0$ we can then choose a representative $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ in the given basis. Likewise for $\dot{M}$. So there are 4 linearly independent pencils. (As many as there are Jacobi fields.)

This means, $n \leq 4$, that the linear system produced by the symplectic invariant is underdetermined and does not compute the volume ratio $|M(0)| / |M(E)|$.

A special case is when $\dot{M}(E) = 0$: in which case the pencil M is "abreast" with respect to U at E. Considering also the pencil M_2 with $M_2(0)=0$ gives immediately

$$M(E)^{\perp}\dot{M}_2(E) = M(0)^T\dot{M}_2(0)$$

$$\Rightarrow \quad \frac{|M(0)|}{|M(E)|} = \frac{|\dot{M}_2(E)|}{|\dot{M}_2(0)|} = \frac{|M_2(E)|}{|\dot{M}_2(0)|}\cdot\frac{|\dot{M}_2(E)|}{|M_2(E)|}$$

$$\cancel{= \frac{|\dot{M}_2(E)|}{|M_2(0)|}\cdot\frac{|M_2(0)|}{|\dot{M}_2(0)|}}$$

The first term is a luminosity distance, missing a redshift factor. Recalling

or 'ang'

$$D^{lum}_{O\to E} = |J_{O\to E}|\cdot|K\cdot U|(0)$$

gives

$$V_{E\to O} = \frac{|K\cdot U|(0)\,|M(0)|}{|K\cdot U|(E)\,|M(E)|} = \frac{D^{lum}_{O\to E}}{|K\cdot U|(E)}\cdot\left|\frac{\dot{M}_2(E)}{M_2(E)}\right|$$

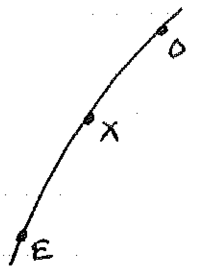
for the induced volume ratio. The second term is a Gauss curvature. For the large geodesic sphere centred at O, the Weingarten at E (in our adopted basis) is

$$S^A{}_C = M_2^{-1\,AB}\,\dot{M}_{2\,BC}$$

the trace gives extrinsic curvature, K, while det gives Gauss: H. So

$$V_{E\to O} = \frac{D_{O\to E}}{|K\cdot U|(E)}\times H_{O\to E}.$$

One might hope to find a determined system for volume ratios induced by arbitrary congruences by making measurements at more that 2 points on T^1. Indeed, consider n pencils and the associated data at $m = 3$ points. Then....



# undetermined matrices $2n \times 2$	# invariants $\frac{1}{2}n(n-1) \times 3$
8	3
12	9
16	18
20	30

So, for 3 points the system is already overdetermined at $n=4$. If M is a congruence with no singular points between E and O, then the light cone congruences M_1, M_2, M_3 vanishing at E, X, O are generically linearly independent (among themselves & with M).

This shows that, in order to determine volume ratios for an arbitrary pencil, one must know all the 'lensing data' of T^1 through X. Knowing $\dot{M}_i(i)$ gives solid angles ~~at~~ subtended at each of the $i = E, X, O$ and the equations determine the luminosity distances $|M_i(j)| / |\dot{M}_i(i)|$ once one of them has been fixed. These are precisely the data that one must choose to compute lensing deflections.