Some notes on ambitwistor spaces

Hadleigh Frost

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We can understand the local structure of ambitwistor space A by describing its tangent spaces. The tangent spaces to A can be identified with Jacobi fields in the following way. Let (M, g) be a complex manifold and consider some geodesic l. It has some tangent vector field K satisfying $\nabla_K K = 0$. We can obtain nearby geodesics by choosing a Jacobi field J, which satisfies $[K, J] = 0$ on l. At a particular point $p \in l$ we can specify the field J by choosing two vectors, $J(p)$ and $\dot{J}(p)$ (this is initial data for the Jacobi equation). The Jacobi fields on l can be regarded as the tangent vectors to the space of geodesics at l. However, the Jacobi fields corresponding to

$$J(p) = K,\ \dot{J}(p) = 0 \qquad \text{and} \qquad J(p) = 0,\ \dot{J}(p) = K$$

merely amount to reparametrisations of l. So the space of geodesics, when it exists, has expected dimension $2 \dim M - 2$. We may represent the tangent vectors by choosing $J(p), \dot{J}(p)$ orthogonal and not parallel to K. There is a natural 1-form, ω, on the tangent space, which resembles the Wronskian. For any two Jacobi fields J_1, J_2, we can, at any point p on l, evaluate

$$\omega(J_1, J_2) = \dot{J}_1 \cdot J_2(p) - J_1 \cdot \dot{J}_2(p). \tag{1}$$

One easily verifies that the right hand side does not depend on our choice of $p \in l$ (by Jacobi's equation). If we constrain l to be null, $K \cdot K = 0$, we obtain a neighbouring null geodesic if $\dot{J}(p)^2 = 0$. So the space of null geodesics has expected dimension $2 \dim M - 3$.

Theorem 1. *(Le Brun) Let M be a complex manifold and $p \in M$ a point. There exists some neighbourhood U of p such that the space of (null) geodesics in U is itself a Hausdorff complex manifold of the expected dimension.*[1]

The space of null geodesics is called ambitwistor space. In section 1 we give examples for flat space times. Our discussion of the relationship with gauge theory begins in section 3, and we conclude by discussing super Yang-Mills in section 6.

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* frost@maths.ox.ac.uk

[1] Le Brun first proved this in his thesis [1] and later elaborated his work in ref. [2].

1 Examples

In principle, A could be given as the symplectic quotient of T^*M by the constraint $P^2 = 0$. However, for flat space we can give instructive global presentations of A using spinors and flag varieties. Let M_n be n-dimensional affine space over $\mathbb{C}$ with the obvious quadratic form. Write A_n for the associated ambitwistor space. We begin with some low dimensional examples.

1.1 Three dimensions

In three dimensions, a null vector k satisfies

$$k_1^2 + k_2^2 + k_3^2 = 0,$$

which is best solved using spinors. Indeed, all null vectors can be written as $k^a = \sigma^a{}_{\alpha\beta}\lambda^\alpha\lambda^\beta$ for some λ^α, unique up to a sign. Here λ^α belongs to the two dimensional spin representation of $so(3)$ and $\sigma^a_{\alpha\beta}$ are the Pauli matrices. A null geodesic is then of the form

$$l = \left\{ x_0^{\alpha\beta} + t\lambda^\alpha\lambda^\beta | t \in \mathbb{C} \right\} \subset M.$$

Notice that l is the solution to the equation

$$\mu^\alpha = x^{\alpha\beta}\lambda_\beta,$$

if we choose $\mu^\alpha = x_0^{\alpha\beta}\lambda_\beta$. The spinor μ^α does not depend on our choice of a base point x_0 on l. It follows that A_3 is given by the pairs $(\mu^\alpha, \lambda_\beta)$, up to a projective scaling. In other words, the ambitwistor space is $A_3 = \mathbb{CP}^3$, which has dimension $2 \times 3 - 3 = 3$, as anticipated.[2] We can equip this with a symplectic form

$$\omega = \mathrm{d}\lambda_\alpha \wedge \mathrm{d}\mu^\alpha. \tag{2}$$

[2]This example is discussed by Le Brun in ref. [3].

The reason we choose this symplectic form—and not, say, the Kähler form on $\mathbb{CP}^3$—is that it is equivalent to equation (1).[3]

1.2 Four dimensions

In four dimensions, a null vector can be decomposed as $k^a\gamma_a{}^{\alpha\dot\alpha} = \tilde\lambda^\alpha\lambda^{\dot\alpha}$, unique up to a scaling. To parameterise the null rays, we can repeat the trick we used in three dimensions. The chiral structure of the spin representations proves to be only a minor complication. The solutions to $\mu^\alpha = x^{\alpha\dot\beta}\lambda_{\dot\beta}$ form a null 2-plane, $\{x_0^{\alpha\dot\beta} + \pi^\alpha\lambda^{\dot\beta} | \forall\pi^\alpha\}$. Likewise, the solutions to $\tilde\mu^{\dot\beta} = x^{\alpha\dot\beta}\tilde\lambda_\alpha$ form a null 2-plane, $\{x_1^{\alpha\dot\beta} + \tilde\lambda^\alpha\tilde\pi^{\dot\beta} | \forall\tilde\pi^{\dot\beta}\}$.[4] It is clear that if these two planes intersect for some choice of $\pi, \tilde\pi$, then they intersect in a null line with tangent vector $\tilde\lambda^\alpha\lambda^{\dot\alpha}$. The two planes intersect if there exist $\pi, \tilde\pi$ such that

$$x_0^{\alpha\dot\beta} - x_1^{\alpha\dot\beta} + \pi^\alpha\lambda^{\dot\beta} - \tilde\lambda^\alpha\tilde\pi^{\dot\beta} = 0.$$

Since the chiral spin representations are only two dimensional, such a π and $\tilde\pi$ exist iff

$$Q = \mu^\alpha\tilde\lambda_\alpha - \lambda_{\dot\alpha}\tilde\mu^{\dot\alpha} = 0. \tag{3}$$

This defines a quadric in $\mathbb{CP}^3 \times \mathbb{CP}^3$. Conversely, every null line can be obtained as the intersection of two such planes. The null ray through $x_0^{\alpha\dot\beta}$ with momentum $\tilde\lambda^\alpha\lambda^{\dot\beta}$ corresponds to the intersection of the planes $(x_0^{\alpha\dot\beta}\lambda_{\dot\beta}, \lambda_{\dot\beta})$ and $(x_0^{\alpha\dot\beta}\tilde\lambda_\alpha, \tilde\lambda_\alpha)$. It follows that A_4 is the quadric $Q = 0$ in $\mathbb{CP}^3 \times \mathbb{CP}^3$, which has dimension $2 \times 4 - 3 = 5$. It has a symplectic form

$$\omega = \mathrm{d}\tilde\lambda_\alpha \wedge \mathrm{d}\mu^\alpha + \mathrm{d}\tilde\mu^{\dot\alpha} \wedge \mathrm{d}\lambda_{\dot\alpha}.$$

This is again identical with equation (1).

1.3 Six dimensions

In six dimensions, the two chiral spin representations are isomorphic and the chiral gamma matrix $\gamma_a^{\alpha\beta}$ is antisymmetric. A null vector has form $\epsilon^{\alpha\beta\gamma\delta}\lambda_\gamma\tilde\lambda_\delta$, though this is not unique. To parameterise the null rays we can again consider $\mu^\alpha = x^{\alpha\beta}\lambda_\beta$, as in three dimensions. However, this has a solution iff $\mu^\alpha\lambda_\alpha = 0$, since $x^{\alpha\beta}$ is skew. The solution is a null 3-plane with tangents of the form $\epsilon^{\alpha\beta\gamma\delta}\pi_\gamma\lambda_\delta$. Likewise, consider a second null 3-plane defined by $\tilde\mu^\alpha = x^{\alpha\beta}\tilde\lambda_\beta$. Again, we must impose $\tilde\mu^\alpha\tilde\lambda_\alpha = 0$. If the two planes intersect, they do so in a null ray with tangent $\epsilon^{\alpha\beta\gamma\delta}\lambda_\gamma\tilde\lambda_\delta$. Following the same argument as in four dimensions, they intersect if

$$Q = \mu^\alpha\tilde\lambda_\alpha + \lambda_\alpha\tilde\mu^\alpha = 0.$$

[3] The translation is straightforward. We identify

$$k^a = \sigma^{a\alpha\beta}\lambda_\alpha\lambda_\beta \qquad \text{and} \qquad \mu^\alpha = x_0^a\sigma^{a\alpha\beta}\lambda_\beta.$$

A variation in the geodesic is specified by $\delta x_0^a = J^a$ and $\delta k^a = \dot J^a$. This is related to the spinors μ^α and λ_α by

$$\delta\mu^a = J^a\sigma^{a\alpha\beta}\lambda_\beta, \qquad \dot J^a = 2\sigma^{a\alpha\beta}\lambda_{(\alpha}\delta\lambda_{\beta)}.$$

It follows that

$$\delta_1\lambda_\alpha\,\delta_2\mu^\alpha = \dot J_1 \cdot J_2,$$

and this shows that equation (2) is the same as equation (1).

[4] Penrose called these α and β planes. They are distinguished in the Grassmannian of null planes by belonging to distinct $SO(4)$ orbits—see below.

So, if $\mu^\alpha\lambda_\alpha = 0$, $\tilde{\mu}^\alpha\tilde{\lambda}_\alpha = 0$, and $Q = 0$, the two planes intersect in a null ray. But the null ray is not uniquely represented by the two null planes. If a null ray through x_0 with tangent $\epsilon^{\alpha\beta\gamma\delta}\lambda_\gamma\tilde{\lambda}_\delta$ is contained in the α-plane $(\mu^\alpha, \lambda_\alpha)$, it is also contained in the plane

$$(\mu^\alpha + t x_0^{\alpha\beta}\tilde{\lambda}_\beta, \lambda_\alpha + t\tilde{\lambda}_\alpha),$$

for all t. With respect to the symplectic form

$$\omega = \mathrm{d}\mu^\alpha \wedge \mathrm{d}\tilde{\lambda}_\alpha + \mathrm{d}\tilde{\mu}^\alpha \wedge \mathrm{d}\lambda_\alpha,$$

this shift in μ^α and λ_α is generated by the Hamiltonian $\tilde{\mu}^\alpha\tilde{\lambda}_\alpha$. Likewise for $\mu^\alpha\lambda_\alpha$. So A_6 is obtained as a symplectic reduction of $\mathbb{CP}^7 \times \mathbb{CP}^7$ by the Hamiltonians $\tilde{\mu}^\alpha\tilde{\lambda}_\alpha$, $\mu^\alpha\lambda_\alpha$, and Q. The first two Hamiltonians reduce the dimension from 14 by two each. Q only reduces the dimension by one since the Hamiltonian vector field associated to Q is just the difference of the two Eulerian vector fields which we have already reduced by when passing to the projective spaces $\mathbb{CP}^7$. The symplectic reduction by these Hamiltonians thus has dimension $14 - 5 = 9$, as expected.

1.4 Eight dimensions

In eight dimensions, the two chiral spin representations have dimension eight, and there are symmetric inner products $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$ on each chiral representation. The gamma matrices satisfy

$$\gamma_{(a}{}^{\alpha\dot{\alpha}}\gamma_{b)\alpha}{}^{\dot{\beta}} = g_{ab}\epsilon^{\dot{\alpha}\dot{\beta}}, \tag{4}$$

where g is the Euclidean metric on M. In fact, analogous identities can be obtained by permuting the roles of the indices. As in lower dimensions, a spinor $\lambda^{\dot{\alpha}}$ defines a plane with tangents $k^a = \gamma^a{}_{\alpha\dot{\alpha}}\pi^\alpha\lambda^{\dot{\alpha}}$. However, this plane is totally null only if $\lambda^{\dot{\alpha}}\lambda_{\dot{\alpha}} = 0$, by equation (4). In this case, $\lambda^{\dot{\alpha}}$ defines a null 4-plane. Such a $\lambda^{\dot{\alpha}}$ is called a pure spinor, and the space of projective pure spinors can be identified with null planes of maximal dimension. (See appendix A.2.) The incidence relation is now

$$\mu_\alpha = x_{\alpha\dot{\alpha}}\lambda^{\dot{\alpha}},$$

and μ_α is a pure spinor, as follows from equation (4). If x_0 is a solution, the space of solutions is a null 4-plane of the form $x_0^a + \gamma^a{}_{\alpha\dot{\alpha}}\pi^\alpha\lambda^{\dot{\alpha}}$, for all π^α. However, the incidence relation has a solution only if

$$\mu_\alpha\lambda_{\dot{\alpha}}\gamma_a{}^{\alpha\dot{\alpha}} = 0. \tag{5}$$

Together with the purity of $\lambda^{\dot{\alpha}}$ and μ_α, this condition means that $(\mu_\alpha, \lambda^{\dot{\alpha}})$ is a pure $SO(10)$ spinor. Likewise, the dual relation $\tilde{w}^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\tilde{\lambda}_\alpha$ gives rise to a null 4-plane, provided that

$$\tilde{\lambda}_\alpha\tilde{w}_{\dot{\alpha}}\gamma_a{}^{\alpha\dot{\alpha}} = 0. \tag{6}$$

If the two null planes intersect, they do so in a null ray. The intersection condition is again the quadric analogous to equation (3). The space of two projective $SO(10)$ pure spinors satisfying $Q = 0$ has dimension 19, as compared with $\dim A_8 = 13$. The reason for the discrepancy is that a null vector k^a is not uniquely written as $\gamma^a_{\alpha\dot{\alpha}}\tilde{\lambda}^\alpha\lambda^{\dot{\alpha}}$. We may add to $\lambda^{\dot{\alpha}}$ any spinor of the form $\gamma_a^{\alpha\dot{\alpha}}l^a\tilde{\lambda}_\alpha$. The space of spinors with this form has dimension 4 or, projectively, 3. With respect to the symplectic form, these translations are generated by equation (5). Likewise, equation (6) generates the analogous translations in $\tilde{\lambda}^\alpha$. The symplectic reduction by these constraints then gives A_8, with dimension 13.

2 The grassmannian presentation

As we saw in the previous section, the spinorial presentation of A_n becomes unwieldy in higher dimensions for two reasons. The first reason is purity. In dimension $2n$, we have been imposing that the pairs (μ, λ) are pure spinors for $SO(2n+2)$. (A pure spinor for $SO(2n+2)$ is called a 'twistor' for M_{2n}.) However, in the spinorial representation, a pure spinor must satisfy many constraints. The second reason is the little group. The representation of a null vector k as $\lambda\tilde{\lambda}$ is not unique in dimensions greater than 2. This is manageable in low dimensions, where the orbits of the little group have low dimensions, but not in higher dimensions. We can solve both problems at once by working directly with the null planes represented by pure spinors. In this section, we will explain this approach in four dimensions, before giving its generalising to all even dimensions. The Grassmannian approach to four dimensions will be used again in section 6 in our discussion of super Yang-Mills. The generalisation to all even dimensions is somewhat technical, and may be skipped.

2.1 Four dimensions

Earlier, we constructed A_4 as a quadric in $\mathbb{CP}^3 \times \mathbb{CP}^3$. We can identify A_4 with the flag variety $F(1,3;4)$ of (1,3)-flags in a four dimensional vector space. To make this clear, introduce homogeneous coordinates v^i, w^i for $\mathbb{CP}^3$. Then A_4 is the quadric $v \cdot w = 0$ in $\mathbb{CP}^3 \times \mathbb{CP}^3$. But $v \cdot w$ is the Plücker relation for the embedding the flag variety $F(1,3;4)$ into the Grassmannians $\mathrm{Gr}(1;4) \times \mathrm{Gr}(3;4)$. This is because $\mathrm{Gr}(1;4)$ is isomorphic to $\mathbb{CP}^3$ where a ray with tangent v^i is sent to the point $[v^i]$. Similarly, $\mathrm{Gr}(3;4)$ is isomorphic to $\mathbb{CP}^3$ where a plane with normal w^i is sent to the point $[w^i]$. Then the ray is contained in the plane iff $v \cdot w = 0$. In fact, we can never obtain the lines $v^i = (\mu^\alpha, 0)$ for finite $x^{\alpha\dot{\alpha}}$ from the incidence relation. So A is only an open subset of $F(1,3;4)$. Alternatively, $F(1,3;4)$ is a natural compactification of A. On the other hand, the Euclidean space M_4 can be identified with an open subset of $Gr(2,4)$, which we might call compactified Euclidean space, M_4^c. This identification leads to the fibrations

$$A_4 \subset F(1,3;4) \leftarrow F(1,2,3;4) \rightarrow Gr(2;4) \supset M_4, \tag{7}$$

which we make use of in sections 4 and 5. Let us now explain the identification of M_4 with a subset of $Gr(2,4)$. Generally, the Grassmannian $Gr(r,d)$, which has dimension $r(d-r)$, may be covered by patches associated to dimension $d-r$ subspaces. This is completely analogous to the covering of $\mathbb{CP}^d$ by inhomogeneous coordinate patches. Given a subspace S of dimension $d-r$, choose a basis $e^1, ..., e^d$ so that S is the span of $e^1, ..., e^{d-r}$. Then we may consider the open subset of planes in $Gr(2,4)$ which are transversal to S. These take the form

$$\begin{bmatrix} \mathbb{1}_{d-r} \\ X \end{bmatrix} \in Gr(r,d),$$

where X is an $r \times d-r$ matrix. The corresponding subspace is the column span of this matrix. In our particular case, take $r=2$ and $d=4$. Then one patch of $Gr(2,4)$ may be written as

$$\begin{bmatrix} \mathbb{1}_2 \\ x^{\alpha\dot{\alpha}} \end{bmatrix},$$

and the $x^{\alpha\dot{\alpha}}$ can be identified here with coordinates on M_4. Indeed, in these coordinates, the construction of the tautological bundle over $Gr(2,4)$ shows that we may identify it with the unprimed chiral spinor bundle on M_4. The Grassmannian $Gr(2,4)$ admits an embedding into $\mathbb{P}\left(\wedge^2\mathbb{C}^4\right) = \mathbb{CP}^5$. If z^i are homogeneous coordinates for $\mathbb{CP}^5$, the Plücker relation is $q(z) = z^1z^2 + z^3z^4 + z^5z^6 = 0$. On the open set $z^6 \neq 0$, we recover z^i/z^6, with $1 \leq i \leq 4$, as coordinates on M_4.[5] So we may view M_4^c as the null rays in $\mathbb{CP}^5$ with respect to q, which we denote as $Gr^0(1,6)$. Likewise, we can consider null planes $Gr^0(2,6)$, and so on. In fact, suppose $Z_1, Z_2 \in Gr^0(1,6)$ are two points in a null plane in $Gr^0(2,6)$. If Z_i corresponds to the point x_i in M_4, then one finds that $q(x_1 - x_2) = 0$.[6] So x_1 and x_2 are null separated in M_4 and $x_1 + t(x_1 - x_2)$ is a null geodesic in M_4. Conversely, given a null geodesic, we obtain such a plane in $Gr^0(2,6)$ by taking two points, X_1 and X_2, on the null ray and taking the null plane associated to $X_1 \wedge X_2$. So $A_4^c = Gr^0(2,6)$. We can present an alternative double fibration

$$A_4 \subset Gr^0(2,6) \leftarrow F^0(1,2;6) \rightarrow Gr^0(1,6) \supset M_4.$$

This looks markedly different from equation (7). Earlier, we identified A_4 with $F(1,3;4)$, whereas now we are identifying it with $Gr^0(2,6)$. The translation between $Gr^0(2,6)$ and $F(1,3;4)$ is spinorial. A plane in $Gr^0(2,6)$ is the same as a simple 2-form $X \wedge Y$ on $V = \mathbb{C}^6$. The even forms in $\bigwedge V$ can be identified with $Spin(6)$. Choosing a decomposition $V = W \oplus W^*$, the irreducible representation of the Clifford algebra is $\bigwedge W$, which has dimension 8. The irreducible chiral spin representations are the even and odd forms in $\bigwedge W$, which we denote by $S_6^\pm$. These are each four dimensional. The associated representations of $Spin(6)$ are denoted γ and $\tilde{\gamma}$. Then $\gamma(X \wedge Y)$ is an endomorphism of S_6^+. Since X and Y are both null, the image of $\gamma(X \wedge Y)$ is a 1-dimensional ray in $S_6^\pm$. Doing the same for $\tilde{\gamma}$, and identifying S_6^- as the dual of S_6^+, we recover the presentation $A_4 = F(1,3;4)$.

2.2 All even dimensions

The approach we have sketched to four dimensions was generalised to higher dimensions by Harnad and Schnider. [4] Since the construction is not quite captured by the example of A_4, we will briefly present their result.

Theorem 2. *(i) The ambitwistor space A_{2n} for M_{2n} is an open subset of the isotropic Grassmannian, $Gr^0(2, 2n+2)$. We may present A_{2n} and M_{2n} as fibrations of an isotropic flag variety,*

$$A_{2n} \subset Gr^0(2, 2n+2) \leftarrow F^0(1,2;2n+2) \rightarrow Gr^0(1, 2n+2) \supset M_{2n}.$$

(ii) For $n > 2$ we have the further identification

$$A_{2n} \subset Gr^0(2, 2n+2) = Gr^0(S_{2n-2}, S_{2n+2}^+),$$

given by (a generalisation of) the Cartan map.[7]

The first part of the theorem proceeds as in our discussion of A_4 with no changes. The second part requires some explanation. The Cartan map identifies maximal null planes with projective pure spinors. Take again V with dimension $2n+2$, and a decomposition $V = W \oplus W^*$ so that the quadratic form on V is

$$q : (W \oplus W^*) \times (W \oplus W^*) \rightarrow \mathbb{C} : (w_1, w_1'), (w_2, w_2') \mapsto w_1'(w_2) + w_2'(w_1).$$

The structure of $Gr^0(n+1, 2n+2)$ depends on $n+1$ mod 2. When $n+1$ is odd, $Gr^0(n+1, 2n+2)$ decomposes into two $SO(2n+2)$ orbits. When $n+1$ is even, $Gr^0(n+1, 2n+2)$ is itself an

[5] These are related to the usual ones by $z^1/z^6 = x_1 + ix_2$, and so on, such that $q(x) = (x^1)^2 + (x^2)^2 + \ldots$.

[6] Since $Z_1 + tZ_2$ is null for all t, we find that, writing $Z_i = (z_i^a, z_i^5, z_i^6)$,

$$q(z_i) + z_i^5 z_i^6 = 0 \qquad \text{and} \qquad 2q(z_1, z_2) + z_1^5 z_2^6 + z_1^6 z_2^5 = 0.$$

Combining these gives $x_1^2 + x_2^2 - 2x_1 \cdot x_2 = 0$, where $x_i^a = z_i^a / z_i^6$.

[7] The generalisation is due to Hanard and Schnider.

$SO(2n+2)$ orbit. To see this, consider the orbits of W and W^* under $SO(2n+2)$. The matrix J exchanging W and W^* has determinants $\det J = n+1$. Let us fix $n+1$ odd. We denote the disjoint orbits of W and W^* by $Gr^+(n+1, 2n+2)$ and $Gr^-(n+1, 2n+2)$. The Cartan map takes a null plane $X \in Gr^+(n+1, 2n+2)$ to the image of $\gamma(X)$, which is a 1-dimensional ray in Λ^+_{2n+2}. By choosing an ordering of the coordinates, we can obtain a canonical ray $S^+_2 \subset S^+_{2n+2}$ which is the spin representation of the subgroup $Spin(2) \subset Spin(2n+2)$ corresponding to the 'first two coordinates'. In fact, let us choose S^+_2 so that $S^+_2 = [\gamma(W)]$. Next, we use that γ is equivariant with respect to $Spin(2n+2)$. The ray $[\gamma(X)] \subset S^+_{2n+2}$ is thus connected to $S^+_2 \subset S^+_{2n+2}$ under $Spin(2n+2)$ since X is connected to W under $SO(2n+2)$. This might appear somewhat tautological! But this is Cartan's result. Cartan (via Chevalley) called the space of rays connected to S^+_2 the projective pure spinors,

$$Gr^0(S^+_2; S^+_{2n+2}).$$

Since Cartan's map identifies this space with the space of $n+1$-dimensional null planes, it has dimension $n(n+1)/2$. (See appendix A.2.) We can generalise Cartan's map to other isotropic Grassmannians besides $Gr^0(n+1, 2n+2)$. We define $Gr^0(S_{2n-2}, S^+_{2n+2})$ to be the 2^{n-1}-dimensional subspaces in S^+_{2n+2} connected to $S^{\pm}_{2n-2}$ by $Spin(2n+2)$. Then, in the same way as for the pure spinors, we can identify a null plane $X \wedge Y \in Gr^0(2, 2n+2)$ with the image of $\gamma(X \wedge Y)$ in S^+_{2n+2}.

3 Double fibrations

There is an intimate relationship between the Yang-Mills equations and the ambitwistor space in four dimensions. We return to this in section 4 after developing some prerequisites. As mentioned earlier, both A_4 and M_4 can be presented as fibrations of $F = F(1, 2, 3; 4)$. F can also be identified with the total space of the projective spinor bundles $(\mathbb{P}S \oplus \mathbb{P}\tilde{S})M$ or, equivalently, with the bundle of null quadrics in the projective tangent space $\mathbb{P}T_0M$. For a general complex manifold M, there is no reason to expect that the null geodesics will define a fibration of $\mathbb{P}T_0M$. However, by Le Brun's theorem, this is always possible locally. So consider any double fibration (of complex analytic spaces)

$$A \xleftarrow{p} F \xrightarrow{q} M. \tag{8}$$

It is sometimes possible to construct vector bundles on M from vector bundles on A.[8] Given a locally free sheaf $\tilde{E}$ on A we can obtain a locally free sheaf on M if $p^*\tilde{E}$, restricted to $q^{-1}(x)$, is free for all points $x \in M$.[9] This means that $E = q_*p^*\tilde{E}$ exists, and E_x is given by the global sections of $p^*\tilde{E}$ restricted to $q^{-1}(x)$. Note also that $\tilde{E}$ and E (as vector bundles) have the same rank. Finally, we can give E a connection ∇ induced by the vertical connection

$$\nabla_v : p^*\tilde{E} \to p^*\tilde{E} \otimes \Omega^1 F/A,$$

where $\Omega^1 F/A$ are the vertical forms with respect to p. In order for this to give a connection on E, we need that $q_*\Omega^1 F/A \simeq \Omega^1 M$ is an isomorphism. If, in addition, the fibres of p are simply connected and the fibres of q are compact, we will call the double fibration good.

Theorem 3. *If the fibration, equation* (8)*, is good, there is an equivalence between vector bundles on A (whose pullbacks are trivial on the fibres of q) and vector bundles with connection on M (whose pullbacks have trivial curvature and monodromy on the fibres of p).*[10]

[8] This construction is a mild generalisation of an idea given by Ward for twistor space.

[9] We will identify a vector bundle E with its locally free sheaf of germs, $\mathcal{O}(E)$, which we denote by E.

[10] The original reference is [5].

Remark. The requirement that the bundles on M have trivial curvature becomes vacuous when the fibres of p are one-dimensional, as they are in the case that A is the space of null rays.

Remark. The construction is not symmetric. We obtain a bundle with connection (E, ∇) on M from a bundle $\tilde{E}$ on A. There is no reason to expect the same to work in the opposite direction. The origin of the asymmetry is that the fibres of q are compact (e.g. quadrics in the projective tangent bundle), whereas the fibres of p are non-compact (e.g. null rays).

Given a good double fibration, the cohomology on A can be related to groups on M. The key fact which gives this result is that $p^*\tilde{E} \otimes \Omega^\bullet F/A$ with differential $\nabla_{F/A}$ is a resolution of the inverse image sheaf $p^{-1}\tilde{E}$. In other words, the following sequence is exact:

$$0 \to p^{-1}\tilde{E} \to p^*\tilde{E} \xrightarrow{\nabla_v} p^*\tilde{E} \otimes \Omega^1 F/A \to 0. \tag{9}$$

The associated exact sequence in cohomology gives

$$... \to H^{k-1}(F, p^{-1}\tilde{E}) \to H^{k-1}(F, p^*\tilde{E}) \to H^{k-1}(F, p^*\tilde{E} \otimes \Omega^1 F/A) \xrightarrow{\delta} H^k(F, p^{-1}\tilde{E}) \to ... \tag{10}$$

In good circumstances, the connecting homomorphism δ gives rise to explicit isomorphisms between $H^k(A, \tilde{E})$ and groups on M, in which case we call δ the Penrose transform. The relation to cohomology on M is given by the Leray spectral sequence, which 'usually' gives[11]

$$H^i(F, S) = H^0(M, R^i q_* S).$$

Likewise, the relation to cohomology on A is given by

$$H^i(F, p^{-1}\tilde{E}) = H^i(A, E),$$

which follows by the same argument, since p^{-1} is adjoint to p_*. Concrete examples of the Penrose transform are given in section 5.

4 Yang-Mills in four dimensions

We now return to four dimensions and very briefly state the following result for Yang-Mills. Suppose that $\tilde{E}$ and (E, ∇) is a pair of bundles on A_4 and M_4 constructed by Theorem 3. The relation to the Yang-Mills equation follows by putting $\tilde{S} = \mathrm{End}(\tilde{E}) \otimes \mathcal{I}^k/\mathcal{I}^{k+1}$ in the long exact sequence, equation (10), where $\mathcal{I}$ is the ideal sheaf of A as a quadric in $Gr(1) \times Gr(3)$.

Theorem 4. *(Manin[12]) The bundle $\tilde{E}$ has a unique extension to the second formal neighbourhood of A. The obstruction to a third order extension is given by $\nabla \star F$, regarded as a class in*

$$H^2(A, End(\tilde{E}) \otimes \mathcal{I}^3/\mathcal{I}^4) \simeq ker(\nabla) \subset H^0(M, \Omega^3_M \otimes End E).$$

The isomorphisms in the theorem follow essentially from equation (10). However, some key identifications—such as $R^2 q_* p^*(\mathcal{I}^1/\mathcal{I}^2) = \Omega^2_M$—rely on the presentation of F as $F(1, 2, 3; 4)$. For this reason, the cohomological construction has not been extended to any more general cases. We note that Witten gave a concrete derivation of this result in ref. [7].

[11]The full statement is that $H^i(M, R^j q_* S)$ abuts to $H^{i+j}(F, S)$. If we have $H^i(M, R^j q_* S) = 0$ for all $i > 0$, then we need do no calculations and the result follows. The sheaves $R^i q_* S$ are modelled on the pre-sheaves $H^i(q^{-1}(U), S)$.

[12]See Buchdahl's paper for a clear exposition. [6] The theorem is given in section 2.4 of his paper.

5 Penrose transform

We conclude our discussion of bosonic ambitwistor space in four dimensions by revisiting the exact sequence given in equation (5). For certain choices of $\tilde{E}$, the associated long exact sequence gives isomorphisms which are known as 'Penrose transforms'. In this section we will compute examples of these isomorphisms. We begin with some definitions. Let $\mathcal{O}(a,b)$ be the sheaf whose sections are functions on A regarded as a quadric in $\mathbb{CP}^3 \times \mathbb{CP}^3$ with homogeneity (a,b). We write $\mathcal{O}(a,b)_F = p^*\mathcal{O}(a,b)$ for the pull back bundle: its sections are functions on $(\mathbb{P}S \oplus \mathbb{P}\tilde{S})M$ with homogeneity (a,b). We may identify $\Omega^1 F/A$ with $\mathcal{O}(1,1)_F$ (a vertical p-form can be identified with a function homogeneous in P of weight p). Notice that the fibres of $F \to M$ are $\mathbb{CP}^1 \oplus \mathbb{CP}^1$. We will make crucial use of the fact that

$$H^0(\mathbb{CP}^1, \mathcal{O}) = \mathbb{C}, \qquad H^k(\mathbb{CP}^1, \mathcal{O}(n)) = 0, \text{ for all } n \geq 0,\ k \geq 1. \tag{11}$$

We also use that $H^0(\mathbb{CP}^1, \mathcal{O}(k))$ is given by the k-homogeneous functions on $\mathbb{CP}^1$ which we represent by symmetric spinor tensors: i.e. $\phi_{\alpha_1 \ldots \alpha_k}$ represents the function $\phi_{\alpha_1 \ldots \alpha_k} \lambda^{\alpha_1} \ldots \lambda^{\alpha_k}$, if λ^α are homogenous coordinates on $\mathbb{P}S$. Given all this, the exact sequence reads

$$0 \to p^{-1}\mathcal{O}(a,b) \to \mathcal{O}(a,b)_F \xrightarrow{\nabla_v} \mathcal{O}(a+1,b+1)_F \to 0.$$

We identify ∇_v with $\lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} \nabla_{\alpha\dot{\alpha}}$. For $a=0, b=0$, we find a long exact sequence

$$\begin{aligned} \ldots \to H^0(M, R^0 q_* \mathcal{O}(a,b)_F) \xrightarrow{\nabla_v} H^0(M, R^0 q_* \mathcal{O}(a+1,b+1)_F) \to H^1(A, \mathcal{O}(a,b)) \\ \to H^0(M, R^1 q_* \mathcal{O}_F(a,b)) \xrightarrow{\nabla_v} H^0(M, R^1 q_* \mathcal{O}_F(a+1,b+1)) \to \end{aligned} \tag{12}$$

We would like to know more about the sheaves $R^i q_* \mathcal{O}_F(a,b)$ appearing in this sequence. These are the sheaves modelled on the pre-sheaves $H^i(q^{-1}(U), \mathcal{O}_F(a,b))$. So, looking at the stalks, we are reduced to computing $H^i(\mathbb{CP}^1 \times \mathbb{CP}^1, \mathcal{O}(a,b)_F)$ which has a Künneth decomposition as

$$H^n(\mathbb{CP}^1 \times \mathbb{CP}^1, \mathcal{O}(a,b)_F) = \bigoplus_{i=0}^{n} H^i(\mathbb{CP}^1, \mathcal{O}(a)) \otimes H^{n-i}(\mathbb{CP}^1, \mathcal{O}(b)).$$

We now set about computing $H^1(A, \mathcal{O}(a,b))$.

5.1 Computation of $H^1(A, \mathcal{O}(a,b))$ for $a \geq 0$ and $b \geq 0$

If $a \geq 0$ and $b \geq 0$, the fourth group appearing in equation (12) vanishes (using the Künneth formula and equation (11)). So, in this case

$$H^1(A, \mathcal{O}(a,b)) = \mathrm{coker} \nabla_v \subset H^0(M, R^0 q_* \mathcal{O}(a+1,b+1)_F).$$

To be explicit, the stalks of the sheaf on the right-hand-side are

$$\left(R^0 q_* \mathcal{O}(a+1,b+1)_F\right)_x = H^0(\mathbb{CP}^1, \mathcal{O}(a+1)) \otimes H^0(\mathbb{CP}^1, \mathcal{O}(b+1)).$$

The sections of these groups are the degree $a+1$ and degree $b+1$ polynomials, respectively. So $H^1(A, \mathcal{O}(a,b))$ is represented by symmetric spinor fields of the form

$$\phi_{(\alpha_1 \ldots \alpha_{a+1})}{}^{(\dot{\beta}_1 \ldots \dot{\beta}_{b+1})}(x),$$

modulo the image of ∇_v, which is given by fields of the form

$$\nabla_{(\alpha_1}{}^{(\dot{\beta}_1} \phi_{\alpha_2 \ldots \alpha_{a+1})}{}^{\dot{\beta}_2 \ldots \dot{\beta}_{b+1})}.$$

When $a=b$, we may instead identify these as $a+1$-forms modulo exact $a+1$-forms. In any case, we conclude that

Proposition. *For $a \geq 0$ and $b \geq 0$,*

$$H^1(A, \mathcal{O}(a,b)) \simeq \frac{\{\textit{spinor fields } \phi_{(\alpha_1 \dots \alpha_{a+1})}{}^{(\dot\beta_1 \dots \dot\beta_{b+1})}\}}{\{\textit{exact spinor fields } \nabla_{(\alpha_1}{}^{(\dot\beta_1}\psi_{\alpha_2 \dots \alpha_{a+1})}{}^{\dot\beta_2 \dots \dot\beta_{b+1})}\}}.$$

It is instructive to explicitly write down the isomorphism appearing in the proposition. This amounts to implementing the long exact sequence homomorphism. We use the fourier decomposition of fields on spacetime and consider a single mode,

$$\phi_k(x) = a_{(\alpha_1 \dots \alpha_{a+1})(\dot\beta_1 \dots \dot\beta_{b+1})}\lambda^{\alpha_1} \dots \lambda^{\alpha_{a+1}}\tilde\lambda^{\dot\beta_1} \dots \tilde\lambda^{\dot\beta_{b+1}} e^{ik\cdot X} \in H^0(M, R^0 q_* \mathcal{O}(a+1, b+1)_F),$$

where a is a constant tensor with respect of X. The pre-image of this mode under ∇_v is clearly

$$\alpha_k(x) = \frac{a(\lambda, \dots, \tilde\lambda, \dots)}{i\lambda \not{k} \tilde\lambda} e^{ik\cdot X}.$$

The connecting homomorphism is then explicitly given by

$$\phi_k(x) \mapsto \bar\partial \alpha_k = 2\pi \bar\delta(k\cdot P) a(\lambda, \dots, \tilde\lambda, \dots) e^{ik\cdot X} \in H^1(A, \mathcal{O}(a,b)),$$

where we have written $\lambda \not{k} \tilde\lambda$ as $k \cdot P$ and we use that

$$\bar\delta(k\cdot P) = \frac{1}{2\pi i}\bar\partial \frac{1}{k\cdot P}.$$

5.2 Computation of $H^1(A, \mathcal{O}(a,b))$ for $a \geq 0$ and $b \leq -2$

When $a \leq -2$ or $b \leq -2$ the second group in equation (12) vanishes because $\mathcal{O}(k)_{\mathbb{CP}^1}$ has no global sections if $k \leq -2$. It follows that $H^1(A, \mathcal{O}(a,b))$ can be written in terms of the fourth and fifth groups in the long exact sequence:

$$H^1(A, \mathcal{O}(a,b)) = \ker \nabla_v \subset H^0(M, R^1 q_* \mathcal{O}_F(a,b)).$$

Assuming that $a \geq 0$, the Künneth decomposition gives

$$\left(R^1 q_* \mathcal{O}(a,b)_F\right)_x = H^0(\mathbb{CP}^1, \mathcal{O}(a)) \otimes H^1(\mathbb{CP}^1, \mathcal{O}(b)).$$

Now we use Serre duality, which gives

$$H^1(\mathbb{CP}^1, \mathcal{O}(-k)) = H^0(\mathbb{CP}^1, \mathcal{O}(-k)^* \otimes K) = H^0(\mathbb{CP}^1, \mathcal{O}(k-2)).$$

Then we can identify the sections of $R^1 q_* \mathcal{O}(a,b)_F$ with symmetric spinor fields having a α indices and $-b-2$ $\dot\beta$ indices. We are interested in the kernel of

$$\nabla_v : H^0(M, R^1 q_* \mathcal{O}_F(a,b)) \to H^0(M, R^1 q_* \mathcal{O}_F(a+1, b+1)).$$

In terms of our representation with spin fields, the group on the right has $a+1$ α indices and $-b-3$ $\dot\beta$ indices. So ∇_v acts as

$$\phi_{(\alpha_1 \dots \alpha_a)(\dot\beta_1 \dots \dot\beta_{-b-2})} \mapsto \nabla_{(\alpha_1}{}^{\dot\beta_1}\phi_{\alpha_2 \dots \alpha_{a+1})(\dot\beta_1 \dots \dot\beta_{-b-2})}.$$

We thus identify $H^1(A, \mathcal{O}(a,b))$, with $a \geq 0$ and $b \leq -2$ with spinor fields satisfying

$$\nabla_{(\alpha_1}{}^{\dot\beta_1}\phi_{\alpha_2 \dots \alpha_{a+1})(\dot\beta_1 \dots \dot\beta_{-b-2})} = 0.$$

This is not a dynamical field equation, since ϕ has $(a+1)(-b-1)$ components, while we are imposing only $a+2$ first order equations on these components. This is to be contrasted with the construction of fields satisfying the wave equation using the analogous methods for the twistorial fibration.

6 Super Yang-Mills

The prominent role of formal neighbourhoods in section 4 suggests supersymmetry.[13] In fact, everything we have said so far can be related to a result for $N = 3$ super Yang-Mills. In four dimensions, the superspace $M_{4|N}$ of type (N, N) is $\Pi(S \oplus S')M$, where S and S' are the primed and unprimed spinor bundles on M_4. We can also present $M_{4|N}$ as a flag variety. Recall that M_4 is an open subset of $Gr(2; 4)$. In place of $\mathbb{C}^4$, we may consider $\mathbb{C}^{4|N}$. (This is the base manifold $\mathbb{C}^4$ with functions $Sym^\bullet\mathbb{C}[x_1, ..., x_4] \otimes Asym^\bullet\mathbb{C}[\theta_1, ..., \theta_N]$.) The superspace $M_{4|N}$ can then be presented as an open subset of the flag variety $F(2|0, 2|N; 4|N)$. To see this, recall that there is a coordinate patch of $Gr(2; 4)$ for which subspaces correspond to the matrices

$$\begin{bmatrix} \mathbb{1}_2 \\ x^{\alpha\dot{\alpha}} \end{bmatrix},$$

where the columns span the subspace. Likewise, a flag of type $2|0 \subset 2|N$ is given by the columns of a matrix

$$\begin{bmatrix} \mathbb{1}_{2\times 2} & 0_{2\times N} \\ x^{\alpha\dot{\alpha}} & \tilde{\theta}^{\dot{\alpha}}_i \\ \theta^\alpha_j & \mathbb{1}_{N\times N} \end{bmatrix}. \tag{13}$$

The first two columns span a $2|0$ subspace, while all columns span a $2|N$ subspace. We identify $M_{4|N}$ as the subset of $F(2|0, 2|N; 4|N)$ given by matrices of this form.[14] A null geodesic through x^a with tangent $\lambda^\alpha\tilde{\lambda}^{\dot{\alpha}}$ lifts to a super null geodesic of dimension $1|2N$. The super null geodesic through $x^a|\theta^\alpha_j, \tilde{\theta}^{\dot{\alpha}}_i$ with tangent $\lambda^\alpha\tilde{\lambda}^{\dot{\alpha}}$ comprises points of the form

$$x^a + tk^a \,|\, \theta^\alpha_j + \phi_j\lambda^\alpha,\, \tilde{\theta}^{\dot{\alpha}}_i + \psi_i\tilde{\lambda}^{\dot{\alpha}}.$$

All this data fits into a flag. The matrix, equation (13), can be enlarged to form

$$\begin{bmatrix} \lambda^\alpha & \mathbb{1}_{2\times 2} & 0_{2\times N} & 0_{2\times 1} \\ x^{\alpha\dot{\alpha}}\tilde{\lambda}_\alpha & x^{\alpha\dot{\alpha}} & \tilde{\theta}^{\dot{\alpha}}_i & \tilde{\lambda}^{\dot{\alpha}} \\ \theta^\alpha_j\lambda_\alpha & \theta^\alpha_j & \mathbb{1}_{N\times N} & \tilde{\theta}^{\dot{\alpha}}_i\tilde{\lambda}_{\dot{\alpha}} \end{bmatrix}.$$

The new column on the left represents a $1|0$ subspace inside the $2|0$ subspace. The new column on the right enlarges the $2|N$ subspace to a $3|N$ subspace. The flag spanned by the first and last columns is not altered if we add $\lambda^\alpha\tilde{\lambda}^{\dot{\alpha}}$ to $x^{\alpha\dot{\alpha}}$. Nor if we add λ^α to any θ^α_j or $\tilde{\lambda}^{\dot{\alpha}}$ to any $\tilde{\theta}^{\dot{\alpha}}_i$. The full flag $1|0 \subset 2|0 \subset 2|N \subset 3|N$ contains too much information, since it determines both a super null ray and a point on the ray. So, forgetting the particular point, we obtain super ambitwistor space, $\tilde{A}_{4|N}$, as the flag variety $F(1|0, 3|N; 4|N)$, which fits into the double fibration

$$F(1|0, 3|N; 4|N) \xleftarrow{p} F(1|0, 2|0, 2|N, 3|N; 4|N) \xrightarrow{q} F(2|0, 2|N; 4|N). \tag{14}$$

The fibres of p project, via q, to super null rays in $M_{4|N}$. By construction the tangent vectors to this distribution of super null rays are

$$\lambda^\alpha\tilde{\lambda}^{\dot{\alpha}}e_{\alpha\dot{\alpha}}, \qquad \lambda^\alpha e^j_\alpha, \qquad \text{and} \qquad \tilde{\lambda}^{\dot{\alpha}}e^i_{\dot{\alpha}},$$

[13]The coordinate rings of formal neighbourhoods can be considered to have ‘odd coordinates’. The textbook example is the formal neighbourhood of a point $Spec(\mathbb{C}) \hookrightarrow Spec(\mathbb{C}[x])$ which is given by $Spec\left(\mathbb{C}[x]/x^2\right)$. Now x is an odd coordinate, squaring to zero, in the coordinate ring of the first neighbourhood of $Spec(\mathbb{C})$.

[14]Notice that we have been lead to introduce N pairs of odd-degree spinors as coordinates, as in the conventional description. This is implicit, for instance, in Chapter 4 of Wess and Bagger’s book. [WessBagger]

where the frame vectors

$$e_{\alpha\dot{\alpha}} = \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}, \quad e^j_\alpha = \frac{\partial}{\partial \theta^\alpha_j} - \tilde{\theta}^{\dot{\alpha}\, j} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}, \quad e^i_{\dot{\alpha}} = \frac{\partial}{\partial \tilde{\theta}^{\dot{\alpha}}_i}$$

satisfy

$$[e^j_\alpha, e^i_{\dot{\alpha}}] = \delta^{ij} e_{\alpha\dot{\alpha}}.$$

Given a vector bundle $\tilde{E}$ on $M_{4|N}$ with connection ∇, we see that $p_* q^* \tilde{E}$ exists if $\tilde{E}$ is integrable on the distribution of super null lines. Let $\nabla^i_\alpha = \nabla(e^i_\alpha)$, and so on. Then $\tilde{E}$ is integrable if,

$$[\lambda^\alpha \nabla^i_\alpha, \tilde{\lambda}^{\dot{\alpha}} \nabla^j_{\dot{\alpha}}] = \delta^{ij} \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} \nabla_{\alpha\dot{\alpha}},$$

while everything else commutes. These are the constraints for $D = 4$ super Yang-Mills. It is known that if $N = 3$, these constraints are precisely equivalent to the field equations. In other words, for $N = 3$ super Yang-Mills, the field equations are equivalent to the integrability of $\tilde{E}$ with respect to the double fibration of super null rays, equation (14). The $N = 3$ is significant for the following reason. The flag space $F(1|0, 3|N; 4|N)$ admits an embedding into $Gr(1|N; 4|N) \times Gr(3|N, 4|N)$. If $v^i|\xi^a_\alpha$ and $w_i|\chi^\alpha_a$ are homogeneous coordinates for these Grassmannians, then the Flag space is the quadric $v^i w_i + \xi \cdot \chi = 0$. But, for $N = 3$ we have $(\xi \cdot \chi)^4 = 0$. It follows that the even-degree functions on $F(1|0, 3|N; 4|N)$ are simply the functions on the third order neighbourhood of $F(1, 3; 4) \subset Gr(1; 4) \times Gr(3; 4)$. In this way, our result for super Yang-Mills implies the Yang-Mills result given in section 4.

A Spinors and purity

In this appendix we review results about the spin representations and pure spinors. We mention the general theory only in passing.

A.1 Some properties by dimension

The properties of the chiral spin representations vary with dimension modulo 4. When $D = 0$ mod 4 there are forms on S^+ and S^-, which we write as $C_{\alpha\beta}$ and $C_{\dot{\alpha}\dot{\beta}}$. When $D = 2$ mod 4, S^+ and S^- are dual to each other, and we write $C_\alpha^{\dot{\beta}}$ and $C_{\dot{\alpha}}^{\beta}$ for the pairing. In $D = 0$ mod 8, $C^{\alpha\beta}$ and $C^{\dot{\alpha}\dot{\beta}}$ are symmetric, while when $D = 4$ mod 8, they are anti-symmetric.[15] Moreover, the p-forms $\gamma^{\alpha\beta}_{(p)}$ are symmetric when $D - 2p = 0$ mod 8 and antisymmetric when $D - 2p = 4$ mod 8. If $D - 2p = 6$ mod 8, $\gamma^{\alpha\dot{\beta}}_{(p)} = -\gamma^{\dot{\beta}\alpha}_{(p)}$ is odd under exchange of the indices, while if $D - 2p = 2$ mod 8, it is even.[16] Combining these observations we comment on two particular cases. First, in $D = 6$,

$$(\gamma^a C)_{\alpha\beta} \quad \text{and} \quad (\gamma^{abc} C)_{\alpha\beta}$$

are the only matrices with this index structure and they are both symmetric. Second, in $D = 10$,

$$(\gamma^a C)_{\alpha\beta} \quad \text{and} \quad (\gamma^{abcde} C)_{\alpha\beta}$$

are symmetric while

$$(\gamma^{abc} C)_{\alpha\beta}$$

[15] See [8] section 6.5.

[16] All this is described in Appendix A of [9].

is skew. Generally, the chiral spin representations for $2n$ dimensions can be constructed as direct sums of the chiral representations for $2n-2$. Finally, one particular case is of interest. Namely, the spin(10) group can be broken to a spin(4) $\times$ spin(6) subgroup, and the chiral 10 dimensional spinors decomposed as tensor products of 4 and 6 dimensional spinors. If we write the 10 dimensional chiral spinor λ^A as $(\lambda_i^\alpha, \lambda^{j\dot\beta})$ under this decomposition, then the corresponding 10 dimensional gamma matrices may be written as[17]

$$\gamma^{\alpha\dot\beta}_{(i\beta)(j\dot\gamma)} = \delta_{ij}\delta^\alpha_\beta\delta^{\dot\beta}_{\dot\gamma}, \quad \gamma^{kl}_{(i\beta)(j\gamma)} = \frac{1}{2}\epsilon_{\beta\gamma}\left(\delta^k_i\delta^l_j - (kl)\right), \quad \text{and} \quad \gamma^{kl}_{(i\dot\beta)(j\dot\gamma)} = \frac{1}{2}\epsilon^{ijkl}\epsilon_{\dot\beta\dot\gamma},$$

where the pair $\alpha\dot\beta$ refers to spinor coordinates on $\mathbb{C}^4$ and kl to coordinates on $\mathbb{C}^6$.

A.2 Pure spinors

The space of pure spinors in $2n$ dimensions is the space of null n-planes in $\mathbb{C}^{2n}$ (with some quadratic form) or, equivalently, the space of complex structures on $\mathbb{C}^{2n}$. From the first definition, we see that the space of pure spinors has dimension $n(n-1)/2$. (We may decompose $\mathbb{C}^{2n} = W \oplus W^*$ so that the quadratic form is written in the form $v^i w_i$ for coordinates (v, w). Then a maximal null plane is given by a choice of skew $n \times n$ matrix X_{ij} such that $w_i = X_{ij}v^j$ is the equation of the plane in V.[18]) The second definition is equivalent to the first. The space of complex structures is $SO(2n)/U(n)$, which we identify with a choice of coordinates $v = Az + B\bar z$ (in terms of some coordinates $z, \bar z$ already given), defined up to $A \sim UA, B \sim UB$ where $UU^* = 1$.[19] The rows of the matrix $[AB]$ define an n-plane in $\mathbb{C}^{2n}$. It is null with respect to the metric $\mathrm{d}z^i\mathrm{d}\bar z^i$. Associated to a null n-plane is an n-form ω. Given this, we may consider the image of $\gamma^{abcde}\omega_{abcde}$, which is one dimensional since the plane is null. Conversely, given a spinor λ^α, we can consider the decomposition of the bilinear $\lambda^\alpha\lambda^\beta$ into p-forms. If the bilinear only has a 5-form component, then λ defines a totally null 5-plane. This turns out to be a correspondence and it was first described by Cartan. Such spinors are called pure. For example, in eight dimensions λ^α is pure if $\lambda^\alpha\lambda^\beta C_{\alpha\beta} = 0$. In ten dimensions, λ^α is pure if $\lambda^\alpha\gamma^a_{\alpha\beta}\lambda^\beta = 0$. In twelve dimensions, $\lambda\gamma^{ab}\lambda = 0$ is sufficient. [12] In general one demands that contractions of the bilinear with the projections

$$P^{\alpha\beta}_{\gamma\delta} = \gamma_{abcd\ldots\gamma}{}^\beta\gamma_\delta{}^{abcd\ldots\gamma}$$

all vanish, except in degree n. We now give some examples of how the purity condition can be related between different dimensions. For instance, a pair of $SO(6)$ spinors $(w_\alpha, \lambda^\alpha)$ is pure as an $SO(8)$ spinor if $w_\alpha\lambda^\alpha = 0$. Likewise, a pair of $SO(8)$ spinors $(w_\alpha, \lambda^{\dot\alpha})$ is pure as an $SO(10)$ spinor if $w_\alpha\lambda_{\dot\alpha}\gamma^{\alpha\dot\alpha}_a = 0$. Finally, using the decomposition of $SO(10)$ spinors described in the previous section, a 10 dimensional chiral spinor λ^A written as $(\lambda^\alpha_i, \lambda^{j\dot\beta})$ is pure if

$$\lambda^{i\alpha}\lambda^{\dot\alpha}_i = 0, \qquad \lambda^{i\alpha}\lambda^j_\alpha = 0, \qquad \lambda^{i\dot\alpha}\lambda^j_{\dot\alpha} = 0.$$

[17] See equation 5.10 of [10].
[18] Appendix A of [9].
[19] An explicit discussion is given in the appendix to [11].

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