Notes on spin fields and their correlators

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In section 1 we will construct the 'spin fields,' which realise the spin representation of $so(D)$ in their OPEs. This is closely related to the construction of current algebras on the worldsheet for the $so(D)$ Lie algebra, and this is where we will begin. We specialise to even dimensions and write $D = 2n$. Let ψ^μ be D free fermions with the OPEs

$$\psi^\mu(z)\psi^\nu(0) \sim \frac{\delta^{\mu\nu}}{z}. \tag{1}$$

One finds that the bilinears $J^{\mu\nu} =: \psi^\mu\psi^\nu :$ define a current algebra for $so(2n)$ of level 1.[1] We can express this result in a manner that mimicks the Cartan basis for $so(2n)$. Let $\epsilon_1, ..., \epsilon_n$ be a basis for the dual $\mathfrak{h}^*$ of a Cartan subalgebra $\mathfrak{h} \subset so(2n)$, so that the roots have weights $\pm\epsilon_i \pm \epsilon_j$. To each of these we associate the following worldsheet fermions

$$f^{\pm\epsilon_i} = \frac{1}{\sqrt{2}}\left(\psi^{2i-1} \mp i\psi^{2i}\right). \tag{2}$$

Notice that the only non trivial OPEs are of the form

$$f^{+\epsilon_i}(z)f^{-\epsilon_i}(0) \sim \frac{1}{z}.$$

Then, associated to every root weight, we can associate the composite field

$$E^{\pm\epsilon_i\pm\epsilon_j} =: f^{\pm\epsilon_i}f^{\pm\epsilon_j} : .$$

These have OPEs which realise the 'raising' and 'lowering' relations in the sense that, e.g.,

$$E^{+\epsilon_i+\epsilon_j}(z)E^{-\epsilon_i+\epsilon_k}(0) \sim \frac{1}{z}E^{+\epsilon_i+\epsilon_k}.$$

On the other hand, we also have, e.g.,

$$E^{+\epsilon_i+\epsilon_j}(z)E^{-\epsilon_i-\epsilon_j}(0) \sim -\frac{1}{z^2} + \frac{1}{z}\left(J^{2i-1,2i} + J^{2j-1,2j}\right).$$

So we identify the worldsheet fields $J^{2i-1,2i}$ with a basis for the Cartan subalgebra, H^i.

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[1] That is to say, they have the following OPEs

$$J^{\mu_1\nu_1}(z)J^{\mu_2\nu_2}(0) \sim k\frac{\delta^{\mu_1\mu_2}\delta^{\nu_1\nu_2}}{z^2} + \frac{\delta^{\mu_1\mu_2}J^{\nu_1\nu_2} - \delta^{\mu_1\nu_2}J^{\nu_1\mu_2} - (\nu_1\mu_2)}{z},$$

with $k = 1$, and the numerator for the $1/z$ term is the usual Lie bracket relation.

Contents

1 Bosonisation and the spin fields

The spin fields are, heuristically, the 'square root' of the fermions $f^{\pm\epsilon_i}$. The reason that they are called spin fields is that their OPEs with ψ^μ realise the spin representation of $so(2n)$. In order to take the 'square root' of the fermions $f^{\pm\epsilon_i}$, we first put them in bosonised form. In place of $f^{\pm\epsilon_i}$ we could write

$$e^{\pm\phi_i},$$

for some free boson fields ϕ_i. By construction, these obey the same OPEs, equation (2), as before. However, they do not have the correct fermionic statistics because $e^{\pm\phi_i}$ commutes with $e^{\pm\phi_j}$ whereas $f^{\pm\epsilon_i}$ anti-commutes with $f^{\pm\epsilon_j}$. To recover the correct statistics we can add a factor of

$$c_i = (-1)^{n_1+\ldots+n_{i-1}} \qquad \text{or} \qquad c_i = e^{\pm i\pi(n_1+\ldots+n_{i-1})}.$$

Here, the n_j are number operators and we can give these explicitly as

$$n_i = \frac{1}{2\pi i}\oint \partial\phi_i.$$

On account of the number operators, $c_i e^{\pm\phi_i}$ and $c_j e^{\pm\phi_j}$ anti-commute. So we identify

$$f^{\pm\epsilon_i} = c_i e^{\pm\phi_i}.$$

We can take the square root of this, but not without an ambiguity in c_i due to the branching of the exponential. The most general choice is

$$S^{A_i} = e^{A_i\phi_i} e^{1\pi\sum_{j=1}^{n} A_i M_{ij} n_j},$$

where A_i is $\pm 1/2$ and M_{ij} is a matrix of signs with all zeroes on and above the diagonal. A basis for the spin representation in D dimensions can be identified with the vectors $A = (A_1, ..., A_n)$ with each A_i being $\pm 1/2$. Then we define the spin fields to be

$$S^A = \prod_{i=1}^{n} S^{A_i}.$$

Now we ask, what is the OPE of ψ^μ with S^A? We can write

$$\psi^{2j} = \frac{i}{\sqrt{2}} \left(f^{+\epsilon_j} - f^{-\epsilon_j} \right) \qquad \text{and} \qquad \psi^{2j-1} = \frac{1}{\sqrt{2}} \left(f^{+\epsilon_j} + f^{-\epsilon_j} \right).$$

So, for instance, the OPE of ψ^{2j-1} with $e^{\pm\phi_i/2}$ gives

$$\psi^{2j-1}(z) e^{\pm\phi_i/2}(0) \sim \frac{1}{\sqrt{2}} \frac{1}{\sqrt{z}} e^{\mp\phi_i/2}.$$

In other words, the effect of ψ^{2j-1} on S^A is to flip a sign in A. Likewise, ψ^{2j} flips a sign in A and gives a factor of i in addition. Reasoning in this way, we conclude that

$$\psi^\mu(z) S^A(0) \sim \frac{1}{\sqrt{2}} \frac{1}{\sqrt{z}} \left(\Gamma^\mu\right)^A_B S^B(0),$$

for some matrices $(\Gamma^\mu)^A_B$ with complex entries. Recalling the $\psi\psi$ OPE, equation (1), we have that

$$\left(\Gamma^\mu \Gamma^\mu + \Gamma^\mu \Gamma^\mu\right)^A_B = 2\delta^{\mu\nu} \delta^A_B,$$

and so these are gamma matrices for $so(2n)$.

1.1 Properties by dimension

Just as the action of ψ^μ flips a sign of A in S^A, the bilinears $\psi^{[\mu}\psi^{\nu]}$ flip two signs. We may thus decompose the S^A into the two chiral representations: S^α and $S^{\dot\alpha}$ where $A = \alpha$ has an even number of minus signs and $A = \dot\alpha$ has an odd number. Since ψ^μ flips one sign, we have

$$\psi^\mu(z) S^\alpha = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{z}} \left(\Gamma^\mu\right)^\alpha_{\dot\beta} S^{\dot\beta}(0). \tag{3}$$

The properties of the chiral representations depend on the dimension mod 4. Consider first $D = 0$ mod 4. Then, in the OPE of $S^\alpha S^\beta$ the most singular term occurs if $\alpha = -\beta$. This is possible because, in $D = 0$ mod 4, α and β are $D/2$ vectors with an even number of minus signs and an even number of plus signs. Using the definition of S^α we find

$$S^\alpha(z) S^\beta(0) \sim z^{-\frac{D}{8}} C^{\alpha\beta} + ...$$

where

$$C^{\alpha\beta} = \delta_{\alpha+\beta} e^{-i\pi\alpha\cdot M\cdot\alpha}.$$

We identify $C^{\alpha\beta}$ with the inner product on the chiral representations, or 'charge conjugation matrix'. This relation could be used to determine the matrix M in accordance with some convention for $C^{\alpha\beta}$. Using this, and performing contractions on equation (3), we find

$$S^\alpha(z) S^{\dot\beta}(0) \sim \frac{1}{\sqrt{2}} z^{-\frac{D-4}{8}} \left(\gamma^\mu \mathcal{C}\right)^{\alpha\dot\beta} \psi_\mu(0) + ...$$

Now we consider $D = 2 \bmod 4$. In this dimension, α, β are $D/2$ vectors with an even number of minus signs and an odd number of plus signs. This means that $\alpha + \beta$ can never be zero, whereas $\alpha + \dot{\beta}$ can. So, proceeding as before, we find

$$S^\alpha(z) S^{\dot{\beta}}(0) \sim\sim z^{-\frac{D}{8}} C^{\alpha\dot{\beta}} + ...,$$

where

$$C^{\alpha\dot{\beta}} = \delta_{\alpha+\dot{\beta}} e^{-i\pi\alpha\cdot M\cdot\alpha}.$$

Moreover, combining this with (3),

$$S^\alpha(z) S^\beta(0) = \frac{1}{\sqrt{2}} z^{-\frac{D-4}{8}} \left(\gamma^\mu \mathcal{C}\right)^{\alpha\beta} \psi_\mu(0) + ...$$

These are the relations first derived in [1]. This concludes our construction of the spin fields and their OPEs.

1.2 Relations between dimensions

The spin fields for all even dimensions are related to each other in the following sense. Given that S^A are the spin fields for $so(2n)$, we may fix $A_n = +1/2$. Then $(A_1, ..., A_{n-1})$ can be regarded as an index for $so(2n-2)$ spin fields. To arrive at the correct OPEs we could set $\phi_n = 0$, which alters all the OPEs by a factor of $z^{1/4}$. However, we cannot remove the number operator n_n from the coefficient,

$$e^{1\pi \sum_{j=1}^{n} A_i M_{ij} n_j}.$$

An example of particular interest is the relation of $D = 10$ to $D = 4$ and $D = 6$. We can write $A = (a_1, a_2, \alpha_1, \alpha_2, \alpha_3)$, such that a and α are indices for $so(4)$ and $so(6)$ spinors. The $D = 10$ spin field may then be partially factorised as

$$S^A = S^a S^\alpha e^{i\pi \sum_{i=3}^{5} \sum_{j=1}^{2} \alpha_i M_{ij} n_j}.$$

We have used that M_{ij} is zero on and above the diagonal. So the $D = 10$ correlators do not factorise as a simple product of $D = 4$ and $D = 6$ correlators. Nevertheless, the lower dimensional results should be related to the $D = 10$ result by dimensional reduction. In particular, fixing $\alpha = (1/2, 1/2, /12)$ and setting ϕ_3, ϕ_4, ϕ_5 to zero, we obtain the $D = 4$ spin fields from the $D = 10$ spin fields. The additional factor of

$$e^{\frac{i\pi}{2} \sum_{i=3}^{5} \sum_{j=1}^{2} M_{ij} n_j}$$

may be absorbed into the definition of the $D = 4$ matrix $C^{\alpha\beta}$.

2 Four dimensions

In this section we study correlators of the spin field CFT associated to dimension four. We find formulas for the correlators of arbitrarily many spin fields and $so(4)$ currents (the $J^{\mu\nu}$ appearing earlier). There are some constraints on the number of spin fields appearing, as we discuss. Perhaps the most interesting aspect of our formulas is that they can be summed in such a way which makes them manifestly Lie theoretic. What this means is explained in section 2.3. Our formulas could be contracted with momenta and polarisations. By themselves, they are insufficient to compute amplitudes. In particular, the formulas are not rational functions of the worldsheet positions, which means that they cannot be used as integrands in the CHY formula. However, they have the attractive property that they are manifestly gauge invariant. We derive our new formulas in section 2.2, following a review in section 2.1 of a previous result.

2.1 Spin field correlators

We first specialise the OPEs derived in section **??** to four dimensions. The OPE of two unprimed spin fields is

$$S_\alpha S_\beta \sim z^{-\frac{1}{4}} \epsilon_{\alpha\beta},$$

and likewise for $S_{\dot\alpha} S_{\dot\beta}$. The OPE $S_\alpha(z) S_{\dot\beta}(0)$ is not singular in $D = 4$, and so the spin correlators factorise into two parts according to chirality. Moreover, Wick's theorem gives immediately that

$$\left\langle \prod_{i=1}^{2M} S_{\alpha_i}(\sigma_i) \right\rangle = \frac{(-1)^M}{2^M M!} \sum_{\rho\in\mathfrak{G}_{2M}} |\rho| \prod_{i=1}^{M} \frac{\epsilon_{\alpha_{\rho(2i-1)}\alpha_{\rho(2i)}}}{\sqrt{\sigma_{\rho(2i-1)\rho(2i)}}}. \tag{4}$$

Here $|\rho|$ is the sign of the permutation ρ. In this way, all the $D = 4$ spin correlators are easily computed. The terms in the sum are in direct correspondence with undirected chords between $2M$ points on a circle. This formula has been vastly simplified by Schlotterer, Hartl, and Stieberger. [2] They claim that equation (4) may be rewritten as

$$\left\langle \prod_{i=1}^{M} S_{\alpha_i}(\sigma_i) S_{\beta_i}(\tau_i) \right\rangle = (-1)^M W^{\frac{1}{2}} \sum_{\rho\in\mathfrak{G}_M} |\rho| \prod_{i=1}^{M} \frac{\epsilon_{\alpha_i\beta_{\rho(i)}}}{\sigma_i - \tau_{\rho(i)}}, \tag{5}$$

where

$$W = \frac{\prod_{i,j=1}^{M} (\sigma_i - \tau_j)}{\prod_{m<n} \sigma_{mn}\tau_{mn}}.$$

Notice that this coincides with the previous formula for $M = 1$. The new formula has only $M!$ terms, each of which is in direct correspondence with a permutation on M points. As SHS point out, the terms appearing here are an over-complete basis since $(1/2, 0)^{\otimes(2M)}$ contains only

$$\frac{(2M)!}{M!(M+1)!}$$

scalars, which is strictly less than $M!$ for $M > 2$. It remains to prove the formula. This can be done inductively, and amounts to showing that their formula correctly factorises near the singularities where two points collide. There are two cases: (i) two σ's or two τ's collide, and (ii) a σ collides with a τ. Let's do case (i). Without loss of generality, consider $\sigma_{12} \to 0$. Fix a permutation ρ. Let ρ' be the permutation obtained by swapping $\rho(1)$ and $\rho(2)$. Notice that $|\rho| = |\rho'|$ since ρ' is obtained from ρ by an even number of flips. The key relation is

$$\epsilon_{\alpha_1,\beta_{\rho(1)}}\epsilon_{\alpha_2,\beta_{\rho(2)}} + \epsilon_{\alpha_1,\beta_{\rho'(1)}}\epsilon_{\alpha_2,\beta_{\rho'(2)}} = \epsilon_{\alpha_1,\alpha_2}\epsilon_{\beta_{\rho(1)},\beta_{\rho(2)}}. \tag{6}$$

It is on the basis of this identity that the formula factorises. We strip away a factor of

$$\langle S_{\alpha_1}(\sigma_1) S_{\alpha_2}(\sigma_2)\rangle = -\frac{\epsilon_{\alpha_1\alpha_2}}{\sqrt{\sigma_{12}}}.$$

This done, what remains is

$$(-1)^{M-1} W_*^{\frac{1}{2}} \sum_{\rho\in\mathfrak{G}_M/\mathbb{Z}_2} |\rho| \frac{\epsilon_{\beta_{\rho(1)}\beta_{\rho(2)}}}{\tau_{\rho(1)\rho(2)}} \prod_{i=3}^{M} \frac{\epsilon_{\alpha_i\beta_{\rho(i)}}}{\sigma_i - \tau_{\rho(i)}},$$

where

$$W_* = \frac{\prod(\sigma_1 - \tau_j)^2}{\prod(\sigma_1 - \sigma_j)^2} \frac{\prod(\tau_{\rho(1)} - \sigma_j)^2}{\prod(\tau_{\rho(1)} - \tau_j)^2} \times W_{red}.$$

To complete the factorisation computation, we must take the limit of, say, $\tau_1 \to \sigma_1$. (We could choose any other τ for this.) Notice that in this limit the factor W_* is only nonvanishing if $\rho(1) = 1$. So we restrict to those ρ satisfying $\rho(1) = 1$. W_{red} then becomes the W factor for the remaining $2M - 2$ spin fields. In this way, we recover the original expression but for $2M - 2$ spin fields. Case (ii) remains. However, this case is more straightforward and the details appear on page 16 of [2]. Together, these cases establish the result by induction.

2.2 Mixed NS and R correlators

We now employ the previous formula, equation (5), to derive new formulas for the correlators of NS and R insertions. Unfixed NS insertions will be constructed from $\psi(z)\psi(w)$ with $z \to w$, fixed NS insertions from $\psi(0)$, and fixed R insertions from $S_\alpha(0)$. As we have seen, the nonvanishing spin correlators in $D = 4$ involve F_{lh} LH fields and F_{rh} RH fields where $F_{lh} = F_{rh} = 0 \bmod 2$. The spin fields can be contracted using

$$\psi^\mu(0) = -\lim_{z\to 0} \frac{1}{\sqrt{2}} \overline{\sigma}^{\mu\,\alpha\dot\alpha} S_\alpha(z) S_{\dot\alpha}(0). \tag{7}$$

Every such contraction decreases F_{lh} and F_{rh} by one. We see that no matter how many contractions we perform, we always have $F_{lh} = F_{rh} \bmod 2$. There are, in general, two cases: $F_{lh} = 0 \bmod 2$ and $F_{lh} = 1 \bmod 2$. For the first case we will consider the correlator of N integrated NS insertions with $F_{lh} = 2m$ left-handed R insertions and $F_{rh} = 2m'$ right-handed. As an example of the second case we will consider N integrated NS insertions with one fixed NS insertion.

2.2.1 $F = 0$ mod 2

Using the results from the previous section, we have the following spin correlator in the general case,

$$\left\langle \prod_{i=1}^{N+M} S_{\alpha_i}(\sigma_i) S_{\beta_i}(\tilde\sigma_i) \prod_{j=1}^{N+M'} S_{\dot\alpha_i}(\tau_i) S_{\dot\beta_i}(\tilde\tau_i) \right\rangle$$
$$= (-1)^M W^{\frac12} \tilde W^{\frac12} \left(\sum |a| \prod_{i=1}^{N+M} \frac{\epsilon_{\alpha_i \beta_{a(i)}}}{\sigma_i - \tilde\sigma_{a(i)}} \right) \left(\sum |b| \prod_{i=1}^{N+M'} \frac{\epsilon_{\dot\alpha_i \dot\beta_{b(i)}}}{\sigma_i - \tilde\sigma_{b(i)}} \right).$$

Using equation (7), we find the following mixed correlator,

$$I = \left\langle \prod_{i=1}^{N} \psi^{\mu_i}(z_i) \psi^{\nu_i}(z_i) \prod_{j=N+1}^{M} S_{\alpha_i}(\sigma_i) S_{\beta_i}(\sigma_i) \prod_{k=N+1}^{M'} S_{\dot\alpha_i}(\tau_i) S_{\dot\beta_i}(\tilde\tau_i) \right\rangle$$
$$= \lim \frac{(-1)^{M-N}}{2^N} W^{\frac12} \tilde W^{\frac12} \sum |a||b| \prod_{i=N+1}^{N+M} \frac{\epsilon_{\alpha_i \beta_{a(i)}}}{\sigma_i - \tilde\sigma_{a(i)}} \prod_{i=N+1}^{N+M'} \frac{\epsilon_{\dot\alpha_i \dot\beta_{b(i)}}}{\sigma_i - \tilde\sigma_{b(i)}} \prod_{i=1}^{N} \frac{\sigma^{\mu_i}{}_{\beta_{a(i)} \dot\beta_{b(i)}} \sigma^{\nu_i \beta_i \dot\beta_i}}{(\sigma_i - \tilde\sigma_{a(i)})(\tau_i - \tilde\sigma_{b(i)})}.$$

Here, we are taking the limit $\sigma_i, \tau_i \to z_i$ and $\tilde\sigma_i, \tilde\tau_i \to z_i$ for all $i = 1, ..., N$. In this limit, we find that the prefactors become

$$W \to \prod_{i=1}^{N} (z_i - z_i) W_{red} \qquad \text{and} \qquad \tilde W \to \prod_{i=1}^{N} (z_i - z_i) \tilde W_{red},$$

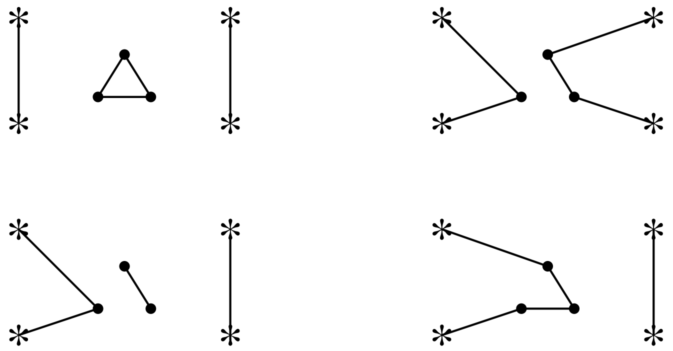

Figure 1: Some topologies for $N = 3$, $M = 2$, $M' = 2$. The asterisks are left and right-handed Ramond insertions. The bullets are NS insertions. The left-handed Ramond insertions appear to the left of the bosons, and the right-handed appear to the right.

where W_{red} is identical to the original expression for W, but restricted to $i = N+1, ..., M$. It follows that the only permutations which give a nonvanishing contribution in this limit are those which set either $a(i) = i$ or $b(i) = i$, but not both, for each $i = 1, ..., N$.[2] For any such nonvanishing permutation, the formula becomes a product of cycles. Some possible cycles are shown in Figure 1, for the case $N = 3$, $M = 2$, $M' = 2$. Finally, by contracting with particle field strengths $F_i^{\mu_i \nu_i}$ we obtain the following cycle formula for the correlator,

$$I = (-1)^{M-N} W_{red}^{\frac{1}{2}} \tilde{W}_{red}^{\frac{1}{2}} \sum |\rho| \prod_{i=1}^{M} \frac{(\epsilon F_{[a_i]})_{\alpha_i \beta_{l(i)}}}{\sigma_{[a_i]}} \prod_{j=1}^{M'} \frac{(\bar{\epsilon} F_{[b_i]})_{\dot{\alpha}_i \dot{\beta}_{r(i)}}}{\sigma_{[b_i]}} \prod_{k=1} \frac{\mathrm{tr}(F_{(c_i)})}{\sigma_{(c_i)}},$$

where ρ has a cycle decomposition as $(a_i)(b_i)(c_i)$. I have adopted a notation so that (a_i) begins with α_i, (b_i) with $\dot{\alpha}_i$ and (c_i) is a closed cycle among the bosons. The summands can be grouped according to the maps $i \mapsto l(i)$ and $i \mapsto r(i)$ which are permutations of the left and right-handed fermions that respect the division into halves. Particularly interesting about these formulas is that the fermions of each chirality are further divided into two halves. The final result does not depend on this arbitrary division. This is a consequence of the Fierz identity, equation (6), as we showed in the proof of Hartl-Schlotterer's spin correlator formula. This gives rise to higher point consequences of the Fierz identities, which we illustrate graphically in figure 2 for the case $N = 1$, $M = 4$, $M' = 0$.

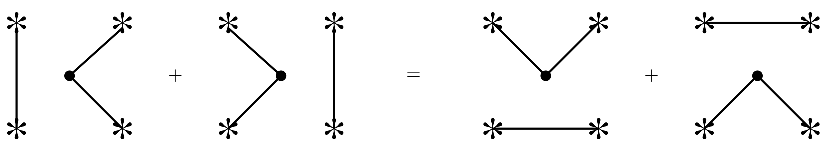

Figure 2: The manifestation of the Fierz identities for the case $N = 1$, $M = 4$, $M' = 0$. On the left, the fermions of each type are split vertically. On the right, the fermions of each type are split horizontally.

2.2.2 $F = 1$ mod 2

The effect of adding one fixed NS insertion is to provide a 'bridge' between the left-handed and right-handed fermions. To see how this works, suppose we combine the two spin fields

[2]Permutations which set $a(i) = b(i) = i$ also vanish since this amounts to performing the contraction $k_i \cdot \epsilon_i$.

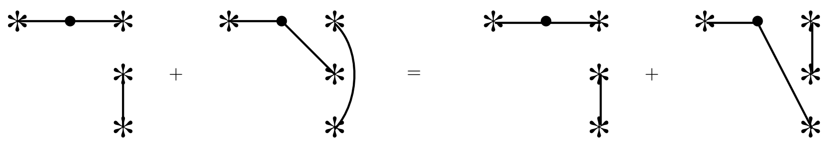

Figure 3: The manifestation of the Fierz identities for the case $N=1$, $M=1$, $M'=3$.

located at σ_1 and τ_1. In the cycle representation of the correlator, we get the following new type of term

$$\frac{(...F_{a_1(2)}F_{a_1(1)}\mathcal{E}F_{b_1(1)}F_{b_1(2)}...)_{\beta_{f(1)}\dot{\beta}_{g(1)}}}{pt^{-1}(z_{f(1)},...,z_{g(1)})},$$

where $\mathcal{E}$ is the polarisation vector of the fixed insertion. The rest of the formula remains largely unchanged. Once again, the higher Fierz identities substantially reduce the number of summands. For instance, of the M' right-handed fermions, one can choose an arbitrary $(M'-1)/2$ subset which is never permitted to join the chain containing $\mathcal{E}$. Any two choices are equivalent, leading to many interesting relations. An example for $N=1$, $M=1$, $M'=3$ is given in figure 3.

2.3 Shuffles

The formulas we have derived in this section involve the field strengths F_i. These are $so(D)$ Lie algebra elements. However, the products of field strengths that appear in the sums are not themselves $so(D)$ Lie algebra elements. In this section, we show that our formulas can be written manifestly in terms of Lie algebra elements. We begin by recalling some basic results about Lie polynomials. Generally, given some symbols $\{X_1,...,X_n\}$, the free Lie algebra is the Lie algebra formally generated by these symbols. In concrete terms, its elements are formed by taking all possible Lie bracketings of the symbols. It is a sub-algebra of the free algebra generated by all possible words formed from the symbols. An element of the free algebra is like a polynomial in non-commuting variables. An element of the free algebra is called a Lie polynomial if it belongs to the free Lie algebra. For instance, consider a homogeneous polynomial of degree s,

$$F=\sum_{\alpha\in\mathfrak{S}_s}c(\alpha)X_{\alpha_1}....X_{\alpha_s}.$$

The following theorem then determines a sufficient condition for F to be a Lie polynomial.

Theorem 1. *(Ree [3]) If the coefficients $c(\alpha)$ obey the shuffle identity, then F is a Lie polynomial.*

For two ordered disjoint set a,b, the set of shuffles is denoted $a\sqcup\!\sqcup\, b$. It comprises all orderings of $a\cup b$ that preserve the ordering of a and b repsectively. Then the 'shuffle identity' referred to in the theorem is the statement that

$$\sum_{\omega\in a\sqcup\!\sqcup b}c(\omega)=0,$$

for all disjoint non-trivial partitions a,b of $\{1,...,s\}$. Given that F is a Lie polynomial, we can apply the following theorem to write F in a way that makes it manifestly Lie. We introduce the following notation for consecutive right-sided Lie bracketings,

$$[1,2,...,s]=[[\,...\,[[[1,2],3],4],\,...\,],s].$$

Then let $[F]$ denote the polynomial obtained from F be replacing every word $X_a...X_b$ with its Lie bracketing, $[X_a, ..., X_b]$.

Theorem 2. *(Dynkin-Specht-Wever) A homogeneous polynomial F of degree s is Lie iff $[F] = sF$.*[3]

Now we return to the new formulas that we have derived in this section. It is shown in appendix A that both the broken Parke-Taylor factors, $pt(\alpha)$, and the Parke-Taylor factors with one position fixed, $PT(*, \alpha)$, satisfy the shuffle identity. Combining the previous two theorems then allows us to write our correlator formulas in a way that is manifestly Lie. For instance, we have encountered the term

$$\frac{(F_{\alpha(1)} \cdots F_{\alpha(k)}\epsilon)_{\alpha\beta}}{\sigma_{[a,\alpha,b]}}.$$

Here α and β are the spinor indices associated to the fermions at σ_a, σ_b. Since we sum over all permutations α, we can use Dynkin-Specht-Wever to replace this term with

$$\frac{1}{k}\frac{\left([F_{\alpha(1)}, \cdots, F_{\alpha(k)}]\epsilon\right)_{\alpha\dot{\beta}}}{\sigma_{[a,\alpha,b]}}. \tag{8}$$

inside the sum. Likewise, consider the term

$$\frac{\mathrm{tr}\left(F_{\alpha(1)} \cdots F_{\alpha(k)}\right)}{\sigma_{(\alpha)}}.$$

By cyclic invariance we can always move, say, F_1 to the first position. Then, in the permutation sum, we may replace this with

$$\frac{k}{k-1}\frac{\mathrm{tr}\left(F_1[F_{\alpha(2)}, \cdots, F_{\alpha(k)}]\right)}{\sigma_{(\alpha)}}. \tag{9}$$

In the first case, equation (8) is a natural pairing $\langle u, Lv\rangle$ where $L \in so(D)$ is the Lie algebra element formed by the field strengths. v is a spinor polarisations in the chiral spinor representation and u is a spinor polarisation in the dual representation. Likewise, equation (9) is also such a pairing except that now $L \in so(D)$ is taken to act of the fundamental representation of $so(D)$ and its dual. Here u and v are given by the polarisation and momentum vectors of F_1.

3 Six dimensions

Having investigated four dimensions in the previous sections, there are several reasons to investigate six. One reason is that we are interested in the spin field CFT for ten dimensions, and $SO(10)$ spinors can be realised as tensor products and direct sums of $SO(4)$ and $SO(6)$ spinors. However, six dimensions is interesting for its own sake. For instance, the bi-adjoint scalar theory is conformally invariant in six dimensions, and its amplitudes can be computed using the ambitwistor string. [5] In this section, we derive formulas for six dimensions which are as exhaustive as those we found in four. We present the new formulas in section 3.2, following a review of the earlier results on spin field correlators in section 3.1

[3] See, e.g., the book by Reutenauer on Free Lie Algebras for this result. [4]

3.1 Spin field correlators

For six dimensions, the spin fields have the following OPEs,

$$S_\alpha(z)S^\beta(0) \sim z^{-\frac{3}{4}}\delta^\beta_\alpha,$$

$$S_\alpha(z)S_\beta(0) \sim \frac{1}{\sqrt{2}}z^{-\frac{1}{4}}\,(\gamma^\mu \mathcal{C})_{\alpha\beta}\;\psi_\mu(0).$$

(This follows directly from section 1.1.) Hartl and Schlotterer [2] present a formula for all non-vanishing spin correlators in $D = 6$. In this section, I comment on its proof. We will use it to obtain new results in the next section. Their formula is

$$\left\langle \prod_{i=1}^N S_{\alpha_i}(\sigma_i) S^{\beta_i}(\tilde{\sigma}_i) \right\rangle = W^{1/4} \sum_\rho |\rho| \prod_{i=1}^N \frac{\delta_{\alpha_i}{}^{\beta_{\rho(i)}}}{\sigma_i - \tilde{\sigma}_{\rho(i)}},$$

where, as before,

$$W = \frac{\prod_{i,j=1}^N (\sigma_i - \tilde{\sigma}_j)}{\prod_{i<j}^N \sigma_{ij}\tilde{\sigma}_{ij}}.$$

The formula clearly holds for $N = 1$, since the relevant OPE in $D = 6$ is

$$S_\alpha(z)S^\beta(0) \sim z^{-\frac{3}{4}}\delta_\alpha{}^\beta.$$

To prove the formula, we need only consider the singularities for $\sigma_i - \tilde{\sigma}_j \to 0$. This is much simpler than the $D = 4$ case. It suffices to consider $i = N$, $j = N$ (due to permutation symmetry). Then the key observation is that, in this singular limit,

$$W \to (\sigma_n - \tilde{\sigma}_n) \frac{\prod_{i,j=1}^{N-1}(\sigma_i - \tilde{\sigma}_j)}{\prod_{i<j}^{N-1} \sigma_{ij}\tilde{\sigma}_{ij}} \frac{\prod_{i=1}^{N-1}(\sigma_i - \tilde{\sigma}_N)(\sigma_N - \tilde{\sigma}_i)}{\prod_{i=1}^{N-1}(\sigma_i - \sigma_N)(\tilde{\sigma}_N - \tilde{\sigma}_i)}.$$

It suffices to notice that

$$\frac{\prod_{i=1}^{N-1}(\sigma_i - \tilde{\sigma}_N)(\sigma_N - \tilde{\sigma}_i)}{\prod_{i=1}^{N-1}(\sigma_i - \sigma_N)(\tilde{\sigma}_N - \tilde{\sigma}_i)} = 1 + \mathcal{O}(\sigma_n - \tilde{\sigma}_n),$$

and this completes the proof.

3.2 Mixed NS and R correlators

We perform contractions on Hartl-Schlotterer's formula to obtain closed formulas for mixed correlators. Compared to our counting in section 2, $D = 6$ is simpler. There is no longer a chiral division of the fermions. We can have any even number of R insertions in the same representation. Finally, we recall that the $D = 6$ OPEs give, for instance,

$$\psi^\mu(0) = +\lim_{z\to 0} 2^{-\frac{3}{2}} z^{\frac{1}{4}} \left(\mathcal{C}^{-1}\bar{\gamma}^\mu\right)^{\alpha\beta} S_\alpha(z)S_\beta(0). \tag{10}$$

We use this to perform the contractions. We will consider the case of N bosons and $2M$ fermions in the same representation. As we will see, it is necessary for M of the bosons to be fixed NS insertions, while the remaining $N - M$ are unfixed. To arrive at our formula for this mixed correlator, we will begin with the Hartl-Schlotterer formula for $2N$ spin fields of each type. Contracting the first $2(N-M)$ of these and taking appropriate limits gives the following formula which I obtained in CFC section 6,

$$I = \left\langle \prod_{i=1}^{N-M} F^i_{\mu_i\nu_i}\psi^{\mu_i}(z_i)\psi^{\nu_i}(z_i)\prod_{j=1}^{2M} S_{\alpha_j}(\sigma_j)S^{\dot\beta_j}(\tilde\sigma_j)\right\rangle$$

$$= 2^{N-M}\left(\frac{\prod_{i,j}(\sigma_i-\tilde\sigma_j)}{\prod_{i<j}\sigma_{ij}\tilde\sigma_{ij}}\right)^{\frac{1}{4}}\sum_{\rho\in\mathfrak{S}_{N+M}}|\rho|\prod_{i=1}^{2M}\frac{\mathcal{C}_{\alpha_i}{}^{\dot\beta_{\rho(i)}}}{\sigma_i-\tilde\sigma_{\rho(i)}}\prod_{i=1}^{N-M}\frac{\left(\mathcal{C}^{-1}F_i\,\mathcal{C}\right)_{\dot\beta_i}{}^{\dot\beta_{\rho(2M+i)}}}{z_i-\tilde\sigma_{\rho(2M+i)}},$$

where it is understood that $\tilde\sigma_{2M+i}=z_i$ for $1\le i\le N-M$. We proceed further by taking the limit in which $\sigma_{2i-1},\sigma_{2i}\to y_i$, and using (10). Contracting with some polarisation data we are, on the LHS, computing

$$I = \left\langle \prod_{i=1}^{N-M} F^i_{\mu_i\nu_i}\psi^{\mu_i}(z_i)\psi^{\nu_i}(z_i)\prod_{j=1}^{2M} S^{\dot\beta_j}(\tilde\sigma_j)\prod_{k=1}^{M}\mathcal{E}_k\cdot\psi(y_k)\right\rangle.$$

We find, on the RHS,[4]

$$\sum_{\rho\in\mathfrak{S}_{N+M}}|\rho|\prod_{i=1}^{N-M}\frac{\left(\mathcal{C}^{-1}F_i\,\mathcal{C}\right)_{\dot\beta_i}{}^{\dot\beta_{\rho(2M+i)}}}{z_i-\tilde\sigma_{\rho(2M+i)}}\prod_{i=1}^{M}\frac{(\mathcal{E}_i\,\mathcal{C})^{\dot\beta_{\rho(2i-1)}\dot\beta_{\rho(2i)}}}{(y_i-\tilde\sigma_{\rho(2i-1)})(y_i-\tilde\sigma_{\rho(2i)})},$$

where

$$W=\frac{\prod_{i=1}^{M}\prod_{j=1}^{2M}(y_i-\tilde\sigma_j)^{1/2}}{\prod_{i\le j}\tilde\sigma_{ij}^{1/4}\prod_{i<j}^{M}y_{ij}}.$$

We have dropped the distracting powers of 2. Fix a permutation ρ. The corresponding summand is a product of two types of cycles: closed cycles containing only bosons and open cycles connecting two fermions. The index structure imposes that

i. every closed cycle contains no fixed bosons,

ii. every open cycle contains an odd number of fixed bosons.

A counting argument then establishes the 'one fixed boson lemma': every fermion chain contains one and only one $\mathcal{E}_i$. Moreover, we claim that a fermion chain only contributes when its $\mathcal{E}_i$ is at the beginning or end of the chain.[5] Given these results, the contributing topologies at low points are shown in figure 4. This leads us also to a cycle representation of the formula,

$$I=(-1)^M W\sum_\rho|\rho|\prod_i\frac{\xi_{l(i)}\mathcal{E}F\ldots\mathcal{C}\xi_{r(i)}}{\sigma_{[a_i]}}\prod_j\frac{\mathrm{tr}F_{(b_i)}}{\sigma_{(b_i)}}.$$

[4]From this calculation we see that we could not have introduced further integrated vertex operators without encountering singularities that cannot be removed—except perhaps by bubbling off some of the $S^{\dot\alpha}$ spin fields.

[5]Our claim is that if $\mathcal{E}$ appears in the middle of a fermion chain in a particular summand this will not contribute to the final result of the permutation sum. We use,

$$\frac{1}{[1234]}+\frac{1}{[1324]}=\frac{\sigma_{14}}{[124][134]}.$$

We also need to recall that $(\gamma\mathcal{C})$ and $(\gamma^2\mathcal{C})$ are anti-symmetric in $D=6$, while $(\gamma^3\mathcal{C})$ is symmetric. In the middle of the fermion chain we have, for some fixed permutation, the term

$$(\ldots)\frac{F_1F_2E_3F_4}{[1234][32]}(\ldots).$$

We can consider also the term obtained from this by swapping 2 and 3. This gives a relative sign and we find, inside the permutation sum,

$$(\ldots)\left[\frac{F_1F_2E_3F_4}{[1234][32]}-\frac{F_1E_3F_2F_4}{[1324][23]}\right](\ldots).$$

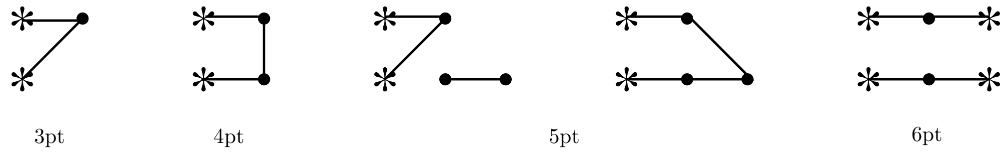

Figure 4: Contributing topologies at low points

Here we have introduced polarisation spinors ξ_i for the fermions. Just as in four dimensions, we can use the properties of the Parke-Taylor factors to infer that the permutation sums give Lie algebra elements inside the traces. Since the results in section 2.3 can easily be adapted to the present case we do not describe this in any further detail.

4 Ten dimensions

In ten dimensions, it is expected that the ambitwistor string gives the correct amplitudes at loop level. In this section, we find formulas for correlators involving two spin fields. These formulas are immediately useful: they can be used to find tree level amplitudes with one fermion line. Moreover, they allow us to directly verify that 1-loop prescription for the ambitwistor string amplitudes. In section 4.1 we review known results for the two spin field correlator. We relate this to an ambitwistor string correlator in section 4.2, and rewrite the integrand in section 4.3 to be manifestly gauge invariant. Finally, in section 4.4, we study the connection to the 1 loop amplitude formulas.

4.1 Two spin fields

We begin by giving the specialisation of our formulas in section 1.1 to ten dimensions. The OPEs are now

$$S_\alpha(z)S_{\dot\beta}(0) \sim z^{-\frac{5}{4}}\mathcal{C}_{\alpha\dot\beta},$$
$$S_\alpha(z)S_\beta(0) \sim \frac{1}{\sqrt{2}}z^{-\frac{3}{4}}\,(\gamma^\mu\mathcal{C})_{\alpha\beta}\;\psi_\mu(0).$$

Härtl and Schlotterer [2] give a formula for the correlator of two spin fields with arbitrarily many fermions ψ^μ in ten dimensions. They gave their result for arbitrary genus, but, for simplicity, we will state and prove it genus zero. Their formula is

$$\left\langle S_\alpha(\sigma_a)S_\beta(\sigma_b)\prod_{i=1}^{2n-1}\psi^{\mu_i}(\sigma_i)\right\rangle = \frac{1}{\sqrt{2}}\frac{1}{(\sigma_{ab})^{\frac{D-4}{8}}}\prod_{i=1}^{2n-1}\frac{1}{\sqrt{\sigma_{ia}\sigma_{ib}}}\times$$
$$\left(\sum_{s=0}^{n-1}\frac{\sigma_{ab}^s}{2^s}\sum_{\rho\in\mathfrak{S}^*_{2n-1}}|\rho|(\Gamma^{\mu_{\rho_1}}...\Gamma^{\mu_{\rho_{2s+1}}}C)_{\alpha\beta}\prod_{j=1}^{n-s-1}\frac{\eta^{\mu_*,\mu_{*+1}}}{\sigma_{*,*+1}}\sigma_{*,a}\sigma_{*+1,b}\right). \quad (11)$$

The spinorial indices can be rearranged such that $F_1E_3F_2F_4 = F_1F_2E_3F_4$ and $F_1F_2E_3F_4 = F_4F_2E_3F_1$. The bracketed expression is then

$$\frac{1}{[32]}\frac{\sigma_{14}}{[124][134]}F_1F_2E_3F_4.$$

Exchanging 1 and 4 does not change the order of the permutation since it takes 6 flips to swap 1 and 4. Then we can add the corresponding summands. That is, we consider

$$X_{(\dots1234\dots)} - X_{(\dots1324\dots)} + X_{(\dots4231\dots)} - X_{(\dots4321\dots)},$$

where X_ρ is the summand associated to ρ. The antisymmetry of $\sigma_{14}/[124][134]$ then shows that this vanishes.

The appearance of an odd number of ψ's and an odd number of Γ matrices will become clear shortly—it is a consequence of the two spin fields having the same chirality. We will treat the opposite chirality case shortly. In the rightmost product, we have introduced an abbreviation: '$*$' is short for $\rho(2(s+j))$ and '$*+1$' for $\rho(2(s+j)+1)$. Finally, the permutation group $\mathfrak{S}^*_{2n-1}$ that appears in the formula is the quotient of $\mathfrak{S}_{2n-1}$ by the subgroup generated by

i. permutations of the indices of the Γ matrices,

ii. permutations of the ordering of the η pairings,

iii. flips of the indices on the factors of η.

The formula may seem unweildy, but it is easily verified. It is certainly correct for $n = 1$ where

$$\langle S_\alpha(\sigma_a) S_\beta(\sigma_b) \psi^\mu(\sigma_1) \rangle = \frac{1}{\sqrt{2}} \frac{1}{\sigma_{ab}^{3/4} \sigma_{1a}^{1/2} \sigma_{1b}^{1/2}} (\gamma^\mu \mathcal{C})_{\alpha\beta} \,.$$

The proof then proceeds by induction. Assuming the formula is correct for $n-1$, the pole at $\sigma_{ij} = 0$ in equation (11) is given by

$$\left\langle S_\alpha(\sigma_a) S_\beta(\sigma_b) \prod_{i=1}^{2n-1} \psi^{\mu_i}(\sigma_i) \right\rangle \simeq \frac{\eta^{\mu_i \mu_j}}{\sigma_{ij}} \left\langle S_\alpha(\sigma_a) S_\beta(\sigma_b) \prod_{k \neq i,j}^{2n-1} \psi^{\mu_k}(\sigma_k) \right\rangle + \ldots$$

Moreover, the pole at $\sigma_{ab} = 0$ is given by

$$\left\langle S_\alpha(\sigma_a) S_\beta(\sigma_b) \prod_{i=1}^{2n-1} \psi^{\mu_i}(\sigma_i) \right\rangle \simeq \frac{1}{\sqrt{2}} (\sigma_{ab})^{-\frac{D-4}{8}} (\gamma_\mu \mathcal{C})_{\alpha\beta} \left\langle \psi^\mu(\sigma_a) \prod_{i=1}^{2n-1} \psi^{\mu_i}(\sigma_i) \right\rangle .$$

To see this, observe that the all ψ correlator follows by Wick's theorem. The formula is

$$\left\langle \prod_{i=1}^{2n} \psi^{\mu_i}(\sigma_i) \right\rangle = \sum_{\rho \in \mathfrak{S}^*_{2n}} \mathrm{sgn}(\rho) \prod_{i=1}^{n} \frac{\eta^{\mu_*, \mu_{*+1}}}{\sigma_{*,*+1}} .$$

To complete the proof, one must also consider the poles at $\sigma_{ai} = 0$ and $\sigma_{bi} = 0$. However, recall the OPE, equation (**??**), of S_α and ψ^μ. To verify equation (11), we must also give a formula for the opposite chirality case. Indeed, taking the $\sigma_{bi} = 0$ pole of (11), we arrive at a conjecture,

$$\left\langle S_\alpha(\sigma_a) S^\beta(\sigma_b) \prod_{i=1}^{2n-2} \psi^{\mu_i}(\sigma_i) \right\rangle = \frac{1}{(\sigma_{ab})^{\frac{D-4}{8}}} \prod_{i=1}^{2n-2} \frac{1}{\sqrt{\sigma_{ia}\sigma_{ib}}} \times$$
$$\left(\sum_{s=0}^{n-1} \frac{\sigma_{ab}^s}{2^s} \sum_{\rho \in \mathfrak{S}^*_{2n-1}} |\rho| (\Gamma^{\mu_{\rho_1}} \ldots \Gamma^{\mu_{\rho_{2s}}} C)_\alpha{}^{\dot\beta} \prod_{j=1}^{n-s-1} \frac{\eta^{\mu_*, \mu_{*+1}}}{\sigma_{*,*+1}} \sigma_{*,a} \sigma_{*+1,b} \right) . \quad (12)$$

The proof by induction then establishes (11) and (12) simultaneously. Assuming, for instance, that (11) holds for $n-1$, we find that (12) has the correct pole

$$\left\langle S_\alpha(\sigma_a) S^\beta(\sigma_b) \prod_{i=1}^{2n-2} \psi^{\mu_i}(\sigma_i) \right\rangle = \frac{1}{\sqrt{2}} (\sigma_{jb})^{-\frac{1}{2}} (\gamma^{\mu_j})^{\beta\gamma} \left\langle S_\alpha(\sigma_a) S_\gamma(\sigma_b) \prod_{i \neq j}^{2n-2} \psi^{\mu_i}(\sigma_i) \right\rangle .$$

The other poles are similar and this concludes the proof of both formulas.

4.2 CHY integrand

To arrive at the CHY integrand computed by the ambitwistor string, we first contract the μ_i indices in equation (11) with polarisations and momenta. To be precise, we consider the product

$$\left(\prod_{i=1}^{n-1} \epsilon_i \cdot \psi(\sigma_{2i-1}) k_i \cdot \psi(\sigma_{2i})\right) \epsilon_{2n-1} \cdot \psi(\sigma_{2n-1}).$$

We also contract $S_\alpha S_\beta$ with spinor polarisations $\xi_a^\alpha \xi_b^\beta$. Next we consider the limit $\sigma_{2i-1,2i} = 0$ (for $1 \leq i \leq n-1$). We do not receive a contribution from singular terms in this limit since $\epsilon_i \cdot k_i = 0$. We must also add the ghost contribution,

$$\left\langle e^{-\phi_a/2} e^{-\phi_b/2} e^{-\phi_n} \right\rangle = \frac{1}{(\sigma_{ab})^{\frac{1}{4}} (\sigma_{na})^{\frac{1}{2}} (\sigma_{bn})^{\frac{1}{2}}}.$$

If we generically denote the polarisations and momenta by a collection of vectors v_i, then the correlator gives the following CHY integrand

$$\begin{aligned} I = & \frac{1}{\sqrt{2}} \frac{1}{\sigma_{ab}} \frac{1}{\sigma_{na}\sigma_{nb}} \prod_{i=1}^{n-1} \frac{1}{\sigma_{2i,a}\sigma_{2i,b}} \times \\ & \lim_{\sigma_{2i-1,2i}\to 0} \left(\sum_{s=0}^{n-1} \frac{\sigma_{ab}^s}{2^s} \sum_{\rho \in \mathfrak{S}^*_{2n-1}} |\rho| (\xi_a \not{\psi}_{\rho(1)} \cdots \not{\psi}_{\rho(2s+1)} C \xi_b) \prod_{j=1}^{n-s-1} \frac{v_* \cdot v_{*+1}}{\sigma_{*,*+1}} \sigma_{*,a}\sigma_{*+1,b} \right). \end{aligned} \quad (13)$$

As we expect for a CHY integrand, I is a rational function of the insertion points. One disadvantage of this presentation of the correlator is that it is not manifestly gauge invariant (under $\epsilon_i \mapsto \epsilon_i + k_i$). Nevertheless, it will be useful to us in section 4.4. In particular, notice that it has a pole at $\sigma_{ab} = 0$.

4.3 A gauge invariant formula

As we discussed in section 2, it may be possible to regard the CHY formula as the unique gauge invariant possibility. It is, therefore, enlightening to consider manifestly gauge invariant presentations of the CHY formula and the closely related integrands that we have been deriving. In this section, we briefly present a manifestly gauge invariant version of the formula, equation (13), presented in the previous section. We do not give its derivation, since it is not important to our main aim in this section—which is to relate the two spin field correlator to loop amplitudes. The manifestly gauge invariant formula is

$$I = \sum_{\alpha \subset \{1 \dots n\}} \frac{(-1)^m}{\sqrt{2}} \frac{(\xi_a F_{\alpha(1)} \cdots F_{\alpha(m)} \gamma^\mu \mathcal{C}\, \xi_b)}{\sigma_{(a,\alpha(1),\dots,\alpha(m),b)}} \frac{\partial}{\partial \epsilon^\mu} \mathrm{Pf}(M^{[bn]}).$$

If so desired, the product of field strengths can be replaced with the total Lie bracket of the field strengths (see section 2.3). Though I derived this formula from the same OPE relations, I do not have a direct algebraic proof demonstrating that it is equal to equation 13. What is certainly true is that both formulas for I have the same pole at $\sigma_{ab} = 0$. Indeed, the pole is given by

$$I = \frac{1}{\sqrt{2}} \frac{1}{\sigma_{ab}} (\xi_a \gamma^\mu \mathcal{C} \xi_b) \frac{\partial}{\partial \epsilon^\mu} \mathrm{Pf}(M^{[bn]}).$$

We mention this here because it will be useful in the following section.

4.4 The forward limit and the degenerating torus

In this section we study the forward limit of the two spin field correlators. In the forward limit we take $k_a + k_b \to 0$. Our motivation for this is that the ambitwistor string integrands for loop amplitudes can be obtained from the forward limits of tree-level correlators. By taking the forward limit of the two Ramond insertions, we will thereby obtain the fermion contribution to a 1-loop amplitude with all external particles being bosons. On its own, the forward limit of the integrand is singular. But, as we will see, this singularity cancels precisely the singularity that arises from bosons running in the loop.

4.4.1 The singular part

He and Yuan studied the solutions to the scattering equations in the forward limit. If we set $k_a + k_b = \tau q$, with $\tau \to 0$, then they found that there are $(n-2)!$ solutions for which $\sigma_{ab} \sim \tau$ and $(n-2)!$ solutions for which $\sigma_{ab} \sim \tau^2$. We call these the singular solutions. The correlator for two spin fields does have a pole in σ_{ab}. This was a key part of the proof we gave in section 4.1. As we mentioned in section 4.3, the pole of the integrand, including ghosts, is

$$I = \frac{1}{\sigma_{ab}} \mathrm{Pf}' \left(M^{[bn]}(b, 1, ..., n) \right),$$

where $M^{[bn]}$ is a CHY matrix with the momenta for b and n removed. The polarisation of b is $\epsilon^\mu = (\xi_a \gamma^\mu \mathcal{C} \xi_b)/\sqrt{2}$. The Pfaffian is linear in the polarisation, and so we can write

$$I = \frac{1}{\sigma_{ab}} \frac{1}{\sqrt{2}} (\xi_a \gamma^\mu \mathcal{C} \xi_b) \frac{\partial}{\partial \epsilon^\mu} \mathrm{Pf}' \left(M^{[bn]}(b, 1, ..., n) \right).$$

To obtain the contribution to the loop amplitude we set $\xi_a = \xi_b$ and sum over a basis of polarisation states. Let us now compare with the contribution from bosons in the loop. To begin, consider $n+2$ bosons. The corresponding integrand is the CHY pfaffian

$$J = \mathrm{Pf}' \left(M^{[an]}(a, b, 1, ..., n) \right).$$

Using the expansion of the Pfaffian we see that it has a pole in σ_{ab} given by

$$J = \frac{1}{\sigma_{ab}} \epsilon_a \cdot \epsilon_b \, \mathrm{Pf}' \left(M^{[b+n,n]}(b, 1, ..., n) \right) + ...$$

Removing $b + n$ from M leaves only the momentum, call it l^μ, associated to b. So we could just as well write this as

$$J = \frac{1}{\sigma_{ab}} \epsilon_a \cdot \epsilon_b \, l^\mu \frac{\partial}{\partial \epsilon^\mu} \mathrm{Pf}'(M^{[bn]}(b, 1, ..., n)).$$

To obtain the loop contribution we set $\epsilon_a = \epsilon_b$ and sum over a basis of polarisation states. We see that the boson contribution cancels the fermion contribution if

$$\frac{1}{\sqrt{2}} \sum_\xi (\xi \gamma^\mu \mathcal{C} \xi) = l^\mu \sum_\epsilon \epsilon \cdot \epsilon. \tag{14}$$

We will assume a normalisation so that this relation holds. Finally, we derive a brief corollary of this. In $D = 10$, the only p-forms that have symmetric index structure ${}^{\alpha\beta}$ are the 1-form and the 5-form:

$$\mathcal{C}^{-1} \bar{\gamma}^{\mu\,\alpha\beta} \qquad \text{and} \qquad \mathcal{C}^{-1} \bar{\gamma}^{5\,\alpha\beta}.$$

(The 3-form has the same index structure, but is antisymmetric.) On these grounds, one must have

$$\sum_h \xi^\alpha \xi^\beta = A_\mu \, \mathcal{C}^{-1} \bar{\gamma}^{\mu\,\alpha\beta} + B_{abcde} \mathcal{C}^{-1} \bar{\gamma}^{abcde\,\alpha\beta},$$

for some p-forms A and B. However, comparing with equation (14), we see that

$$A^\mu = \frac{\sqrt{2}}{2^{D/2-1}} l^\mu \sum_\epsilon \epsilon \cdot \epsilon.$$

If we normalise the boson polarisation sum so that $\sum \epsilon \cdot \epsilon = 2^{-(D-3)/2}$ we find that

$$\sum_h \xi^\alpha \xi^\beta = l_\mu \, \mathcal{C}^{-1} \bar{\gamma}^{\mu\,\alpha\beta} + B_{abcde} \mathcal{C}^{-1} \bar{\gamma}^{abcde\,\alpha\beta}. \tag{15}$$

We will use this in the following section.

4.4.2 Relation to degenerating torus

We expect that the forward limit of our integrand I is related to the integrand one would obtain from the ambitwistor string at genus 1. (Following, in particular, reference [6].) Taking our expression equation (13) for the integrand, we find that

$$\sum_\xi I = \lim_{\sigma_{2i-1,2i} \to 0} \frac{1}{\sqrt{2}} \frac{1}{\sigma_{ab}} \frac{1}{\sigma_{na}\sigma_{nb}} \prod_{i=1}^{2n-2} \frac{1}{\sqrt{\sigma_{ia}\sigma_{ib}}} \times \\ \left(\sum_{s=0}^{n-1} \frac{\sigma_{ab}^s}{2^s} \sum_{\rho \in \mathfrak{S}^*_{2n-1}} |\rho| \mathrm{tr}(\not{v}_{\rho(1)} ... \not{v}_{\rho(2s+1)} \not{l}) \prod_{j=1}^{n-s-1} \frac{v_* \cdot v_{*+1}}{\sigma_{*,*+1}} \sigma_{*,a}\sigma_{*+1,b} \right) + ... \tag{16}$$

We have used equation (15) and moved the pre-factor back inside the limit. There is a second term involving the 5-form B_{abcde} which appears in equation (15). We focus on the first term for now. Following Roehrig and Skinner (who did a similar case), the summands in this formula can be identified with Pfaffians. We write

$$\not{v}_{\rho(1)} \not{v}_{\rho(2)} ... \not{v}_{\rho(2s+1)} \not{l} = \frac{1}{2}(1 + \Gamma_{11}) \not{v}_{\rho(1)} \not{v}_{\rho(2)} ... \not{v}_{\rho(2s+1)} \not{l},$$

where the $\not{v}$ appearing on the right hand side are formed using the full gamma matrices. The factor of $(1 + \Gamma_{11})/2$ is the projection onto the chiral part. The trace can be written in terms of pfaffians as

$$\mathrm{tr}(\not{v}_1 .... \not{v}_{2s+2}) = 2^5 \mathrm{Pf}(V) \qquad \mathrm{tr}(\not{v}_1 .... \not{v}_{2s+2} \Gamma_{11}) = \frac{2^5}{9!!} \int \mathrm{d}^{10}\Psi \mathrm{Pf}(A),$$

where

$$V_{ij} = v_i \cdot v_j \mathrm{sgn}(i-j) \qquad A_{ij} = v_i \cdot v_j \mathrm{sgn}(i-j) + v_i \cdot \Psi v_j \cdot \Psi.$$

The integral over Ψ is a Grassmann integral, with Ψ odd. Rearranging equation (16) we have

$$\sum_\xi I = \lim_{\sigma_{2i-1,2i} \to 0} \frac{1}{\sqrt{2}} 2^{-(n-1)} \frac{1}{\sigma_{ab}^2} \frac{\sigma_{ab}}{\sigma_{na}\sigma_{nb}} \prod_{i=1}^{2n-2} \sqrt{\frac{\sigma_{ab}}{\sigma_{ia}\sigma_{ib}}} \times \\ \left(\sum_{s=0}^{n-1} \sum_{\rho \in \mathfrak{S}^*_{2n-1}} |\rho| \mathrm{tr}(\not{v}_{\rho(1)} ... \not{v}_{\rho(2s+1)} \not{l}) \prod_{j=1}^{n-s-1} 2 v_* \cdot v_{*+1} \frac{\sigma_{*,a}\sigma_{*+1,b}}{\sigma_{*,*+1}\sigma_{ab}} \right) + ... \tag{17}$$

The sum given here in parantheses may be recast as a sum over pfaffians. To do this, we will write $l = v_{2n}$. Then

$$(...) = \sum_{\alpha\subset\{1,...,2n\}} \text{sgn}(\alpha,\alpha^c) 2^5 \text{Pf}(V_{ij})_{ij\in\alpha}\text{Pf}(M_{ij})_{ij\in\alpha^c} + ...,$$

where the elipsis is the analogous term involving the matrix A. The pfaffians combine according to

$$\sum_{\alpha\subset\{1,...,2n\}} \text{sgn}(\alpha,\alpha^c)\text{Pf}(V_{ij})_{ij\in\alpha}\text{Pf}(M_{ij})_{ij\in\alpha^c} = \text{Pf}(V+M),$$

which follows from the definition of the Pfaffian, expanding the right hand side. The entire expression may thus be written as

$$\sum_\xi I = \lim_{\sigma_{2i-1,2i}\to 0} \frac{2^3}{\sqrt{2}}\frac{1}{\sigma_{ab}^2}\sqrt{\frac{\sigma_{ab}}{\sigma_{na}\sigma_{nb}}}\left(\text{Pf}(X) + \frac{1}{9!!}\int d^{10}\Psi \text{Pf}(Y)\right) + \text{five-form term},$$

where

$$X_{ij} = \sqrt{\frac{\sigma_{ab}}{\sigma_{ia}\sigma_{ib}}}\sqrt{\frac{\sigma_{ab}}{\sigma_{ja}\sigma_{jb}}}\left(\frac{1}{2}\text{sgn}(i-j) + \frac{\sigma_{ia}\sigma_{jb}}{\sigma_{ij}\sigma_{ab}}\right) v_i\cdot v_j.$$

and Y is the analogous matrix formed from A. (Notice that we have absorbed a factor of $2^{-(n+1)}$ into the matrix entires.) We can rearrange X_{ij} so that

$$X_{ij} = v_i \cdot v_j S(i,j),$$

where

$$S(i,j) = \frac{1}{\sigma_{ij}}\frac{1}{2}\left(\sqrt{\frac{\sigma_{ib}\sigma_{ja}}{\sigma_{ia}\sigma_{jb}}} + \sqrt{\frac{\sigma_{jb}\sigma_{ia}}{\sigma_{ja}\sigma_{ib}}}\right).$$

In this form, $S(i,j)$ is the Szegö kernel for the torus in the degenerating limit (for two of the four possible spin structures). See for example equation (2.33) of [6], where the spin structures that give rise to this limit are labelled $\alpha = 1, 2$—where these are associated to the Ramond contribution at 1-loop. In this way, $\sum_\xi I$ (less the five-form term) can be understood to arise from the Ramond sector of the ambitwistor string at one loop, in the degenerating limit. (See especially section 3.2 of [6] for a discussion of the Ramond contribution to a 1-loop amplitude with all external states bosonic.)

A Shuffles and Lie polynomials

In this section we prove two results which are first used in section 2.3 in the main text. The first proof is merely a translation from that which appears following pg. 40 in reference [7]. The second proof adapts the idea to a new case that we need in the main text.

Theorem 3. *The broken Parke-Taylor factors satisfy the shuffle identity,*

$$\sum_{\alpha\sqcup\beta} pt(\omega) = 0,$$

where it is understood that the sum is over all $\omega \in \alpha \sqcup \beta$.

Proof. The result holds for $|\omega| = 2$ and $|\omega| = 3$. We proceed by induction. The key observation is that $\alpha \sqcup \beta$ can be decomposed into two sets. This is because every element of $\alpha \sqcup \beta$ must end with either the last letter in α or the last letter in β. Let $|\alpha| = a$ and $|\beta| = b$ be the orders of α and β. Then

$$\alpha \sqcup\!\sqcup\, \beta = (\alpha \sqcup\!\sqcup\, \beta_{-1}, \beta_b) \cup (\alpha_{-1} \sqcup\!\sqcup\, \beta, \alpha_a).$$

We can iterate this decomposition to obtain an expansion

$$\sum_{\alpha \sqcup\!\sqcup \beta} pt(\omega) = \sum_{(\alpha \sqcup\!\sqcup \beta_{-2}, \beta_{b-1}, \beta_b)} pt(\omega) + \sum_{(\alpha_{-1} \sqcup\!\sqcup \beta_{-1}, \alpha_a, \beta_b)} pt(\omega) + \sum_{(\alpha_{-1} \sqcup\!\sqcup \beta_{-1}, \beta_b, \alpha_a)} pt(\omega) + \sum_{(\alpha_{-2} \sqcup\!\sqcup \beta, \alpha_{a-1}, \alpha_a)} pt(\omega).$$

Using the inductive hypothesis we can combine these into two sums. This is because, for example,

$$(\alpha \sqcup\!\sqcup\, \beta_{-2}, \beta_{b-1}) \cup (\alpha_{-1} \sqcup\!\sqcup\, \beta_{-2}, \alpha_a) = (\alpha \sqcup\!\sqcup\, \beta_{-1}).$$

We also use

$$pt(1, ..., m) = pt(1, ..., m-1) pt(m-1, m).$$

Then

$$\sum_{\alpha \sqcup\!\sqcup \beta} pt(\omega) = \left(\sum_{(\alpha_{-1} \sqcup\!\sqcup \beta_{-1}, \alpha_a)} pt(\gamma) \right) (pt(\alpha_a, \beta_b) - pt(\beta_{b-1}, \beta_b)) + \left(\sum_{(\alpha_{-1} \sqcup\!\sqcup \beta_{-1}, \beta_b)} pt(\gamma) \right) (pt(\beta_b, \alpha_a) - pt(\alpha_{a-1}, \alpha_a)).$$

Repeating this expansion, and then combining terms using the inductive hypothesis we obtain

$$\sum_{\alpha \sqcup\!\sqcup \beta} pt(\omega) = \left(\sum_{(\alpha_{-2} \sqcup\!\sqcup \beta_{-1}, \alpha_{a-1})} pt(\gamma) \right) C,$$

where

$$C = pt(\alpha_a, \alpha_{a-1}, \beta_b) + pt(\alpha_{a-1}, \alpha_a, \beta_b) + pt(\alpha_{a-1}, \beta_b, \alpha_a) - pt(\alpha_a, \beta_{b-1}, \beta_b) - pt(\beta_{b-1}, \alpha_a, \beta_b) - pt(\beta_{b-1}, \beta_b, \alpha_a).$$

The shuffle identity for $|\omega| = 3$ then shows that $C = 0$. □

Theorem 4. *The Parke-Taylor factors $PT(*, 1, 2, ..., n)$ with one point fixed obey the shuffle identity.*

Proof. The key relation is that

$$PT(*, 1, ..., n) = PT(*, 1, ..., n-1) \Omega_{n-1,n},$$

where

$$\Omega_{n-1,n} = \frac{\sigma_{*,n-1}}{\sigma_{*,n} \sigma_{n-1,n}}.$$

Just as in the previous proof, one expands the shuffle sum to find

$$\sum_{\alpha \sqcup\!\sqcup \beta} PT(*, \omega) = \left(\sum_{(\alpha_{-2} \sqcup\!\sqcup \beta_{-1}, \alpha_{a-1})} PT(*, \gamma) \right) C,$$

where C has the form

$$C = (\Omega_{ca}\Omega_{ab} + \Omega_{cb}\Omega_{ba} - \Omega_{cb}\Omega_{ca}) - (\Omega_{da}\Omega_{ab} - \Omega_{da}\Omega_{db} + \Omega_{db}\Omega_{ba}).$$

An explicit expansion shows that both of the bracketed terms appearing here vanish independently. □

References

[1] Daniel Friedan, Stephen Shenker, and Emil Martinec. Covariant quantization of superstrings. *Physics Letters B*, 160(1):55–61, October 1985.

[2] D. Haertl, O. Schlotterer, and S. Stieberger. Higher Point Spin Field Correlators in D=4 Superstring Theory. *Nuclear Physics B*, 834(1-2):163–221, July 2010. arXiv: 0911.5168.

[3] Rimhak Ree. Lie Elements and an Algebra Associated With Shuffles. *Annals of Mathematics*, 68(2):210–220, 1958.

[4] Christophe Reutenauer. *Free Lie Algebras*. London Mathematical Society Monographs. Oxford University Press, Oxford, New York, May 1993.

[5] Tim Adamo, Ricardo Monteiro, and Miguel F. Paulos. Space-time CFTs from the Riemann sphere. *Journal of High Energy Physics*, 2017(8), August 2017. arXiv: 1703.04589.

[6] Yvonne Geyer, Lionel Mason, Ricardo Monteiro, and Piotr Tourkine. One-loop amplitudes on the Riemann sphere. *Journal of High Energy Physics*, 2016(3), March 2016. arXiv: 1511.06315.

[7] Jacky Cresson. Calcul Moulien. *arXiv:math/0509548*, September 2005. arXiv: math/0509548.