B5, Schwarzschild I

Due end of MT3 (before Sunday!)

Merton College

n.b. Geodesics can be found by extremising the action

$$S = \int g_{ab}\, \dot{x}^a \dot{x}^b \, d\tau .$$

Also, if λ is proper time, by definition

$$g_{ab}\left(\frac{dx^a}{d\lambda}\right)\left(\frac{dx^b}{d\lambda}\right) = +1,\ 0,\ \text{or}\ -1$$

depending on whether the geodesic is spacelike, null, or timelike.

1. (Planet orbiting a star)
Consider a massive particle (or planet, say) moving in the equatorial $\theta = \pi/2$ plane of the Schwarzschild metric. Show that

$$E = \left(1 - \frac{r_s}{r}\right)\frac{dt}{d\lambda}$$

and

$$J = r^2 \frac{d\phi}{d\lambda}$$

are constants of the motion. λ is proper time.
Show that

$$\left(\frac{dr}{d\lambda}\right)^2 + V(r) = E^2$$

for some function $V(r)$. How does $V(r)$ differ from the effective radial potential in the Newtonian Kepler problem? For what values of J does $V(r)$ have a stable minimum?

Now, find ~~the~~ a second order equation for r by extremising

$$S = \int g_{ab}\, \dot{x}^a \dot{x}^b \, d\tau .$$

~~each~~ ~~use~~ ~~this~~ Then consider a circular orbit at radius r_0. Show that

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{r_s}{2 r_0^3} .$$

2. (Light orbiting a black hole)
Consider a photon moving in the equatorial plane of the Schwarzschild metric. As in Q1, determine the effective radial potential, $V(r)$.

Are there any circular orbits? Are they stable?

3. (Light moving radially)
Consider a photon outside ($r > r_s$) a Schwarzschild black hole and moving radially (θ, ϕ are constant).

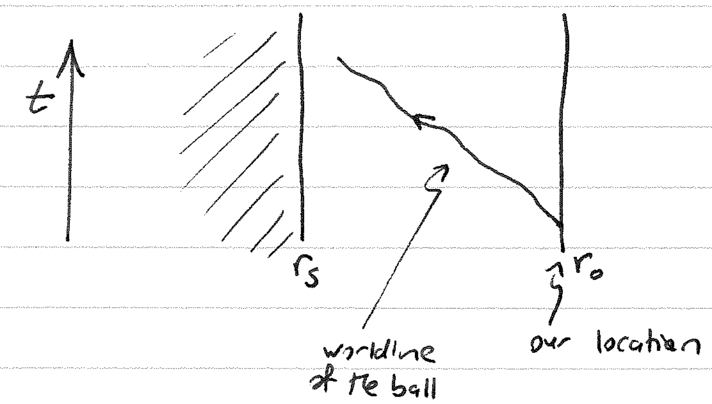
Find the function $f(r)$ so that

$$f(r) - t$$

is constant on the null geodesic.

4. (Throwing a ball)

Let us hover at fixed r_0, θ, ϕ outside a Schwarzschild black hole. We throw a ball towards the hole, which moves radially.



We watch the ball.

How long does it take for the ball to fall into the horizon at $r = r_s$.

5. (Glowing hydrogen)

Suppose some warm hydrogen gas orbits a Schwarzschild black hole at roughly $r = 4r_s$. It emits light at frequency ν_H. Work out the 'speed'

$$s = \frac{d\,r\phi}{dt},$$

which is the speed relative to a stationary observer. We look at the hydrogen from a large distance.

What range of frequencies do we see?