B5 Gravitational Radiation I

Due end HT5

Merton College

1. (Merging Black Holes)
Using the Schwarzschild metric, work out the surface area of the horizon of a Schwarzschild black hole.

If two well-separated Black Holes with surface areas A_1 and A_2 merge, Hawking argues that the new black hole has surface area

$$A_{new} \geq A_1 + A_2 .$$

Find an upper bound for the total ~~amount~~ energy of gravitational radiation released when two black holes of mass m_1 and m_2 merge.

2. (Linearised Field Equations)
Einstein's equation is $G_{ab} = 8\pi G T_{ab}$. Now put

$$g_{ab} = \eta_{ab} + h_{ab} \quad \text{(i.e. "g is nearly flat")}$$

where h_{ab} is small. To first order in h_{ab}, show that

$$R_{abcd} = \frac{1}{2}\left(h_{ac,bd} + h_{bd,ac} - h_{ad,bc} - h_{bc,ad}\right).$$

What is Einstein's equation to first order in h_{ab}?

To simplify your answer, write everything in terms of

$$\bar{h}_{ab} = h_{ab} - \tfrac{1}{2}\eta_{ab} h. \qquad (h = \eta^{ab} h_{ab})$$

Show that Einstein's equation to first order in $\bar{h}_{ab}$ becomes

$$\partial^2 \bar{h}_{ab} = 16\pi G\, T_{ab} \qquad (*)$$

if you assume that $\partial^a \bar{h}_{ab} = 0$.

2*. (Optional: Gauge freedom)
Consider a coordinate transformation

$$x'^a = x^a + f^a(x).$$

Working to first order in f^a and h_{ab}, show that

$$h'_{ab} = \partial_a f_b + \partial_b f_a + h_{ab}.$$

Does there exist a unique choice of $f^a(x)$ so that $\partial^a \bar{h}'_{ab} = 0$? How does this bear on Q2?

3. (Gravitational Waves)
Find a gravitational wave solution $h_{ab}(x)$ to the linearised Einstein equation $(*)$ in a vacuum $(T_{ab} = 0)$, that has momentum k^a. (I.E. $\partial_c h_{ab}(x) = i k_c \cdot h_{ab}(x)$.)

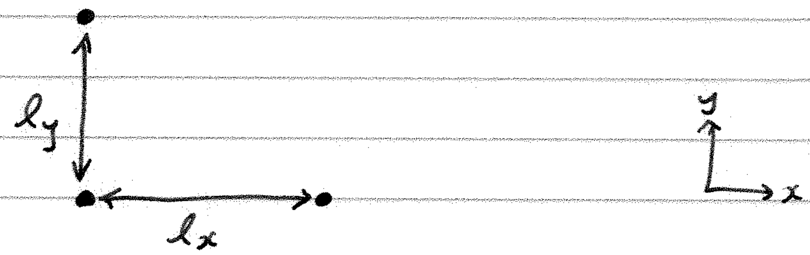
What is the mass of a gravitational wave?

4. (Measuring Gravitational Waves)

Suppose $h_{00} = 0$ and $h_{0i} = 0$ $(i = 1,2,3)$. What are the only non-zero Christoffel symbols? (To first order in h_{ab}.)

Show that the worldlines "x^i = constants" are geodesics (to first order in h_{ab}).

Consider three such worldlines at positions $(0,0,0)$ and $(s,0,0)$ and $(0,s,0)$.



A gravitational wave $\bar{h}_{ab} = A_{ab}\, e^{ik\cdot x}$ with

$$k^a = (1,0,0,1) \quad \text{and} \quad A_{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

passes through. What are the proper distances l_x and l_y as a function of time?

These oscillations could be measured using interferometry. What would be a good frequency of light to use?

5. (Gravitational Waves Production)

How are gravitational waves produced? Earlier, we found equation (*). Solve this equation.

Suppose T_{ab} is nonzero only near $\underline{x} = 0$. To leading order in $|\underline{x}| = r$, show that

$$\bar{h}_{ij}(t, \underline{x}) = \frac{4G}{r} \int d^3y \; T_{ij}(t + |\underline{x} - \underline{y}|, \underline{y}).$$

If you want to, prove that

$$2T_{ij} = (T_{00}\, x^i x^j)_{,00} + \text{total spatial divergence}$$

(by expanding $(T^{ab} x^i x^j)_{,ab}$ out). Infer that

$$\bar{h}_{ij}(t, \underline{x}) = \frac{2G}{r} \ddot{I}_{ij}$$

where I_{ij} is the 2^{nd} moment of T_{00}.