B5 Cosmology I

Due end HT 7

Merton College

Geometry

1. (The Metric)

The Friedmann metric is

$$g = -dt^2 + a(t)^2 \gamma$$

where $a(t)$ is called the 'scale factor' and γ is a 3-metric of a 3-sphere, 3-hyperboloid, or flat space. Consider the 3-sphere

$$w^2 + x_i x_i = 1$$

in $\mathbb{R}^4$. Show that the metric on the sphere, induced by $dw^2 + dx_i dx_i$, is

$$g = \left(\delta_{ij} + \frac{x_i x_j}{1 - \underline{x}^2} \right) dx^i dx^j .$$

What is the induced metric on the 3-hyperboloid

$$-w^2 + x_i x_i = 1 \, ?$$

Convert to spherical coords and show that the Friedman metric can be written as

$$g = -dt^2 + a^2 \left(\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right)$$

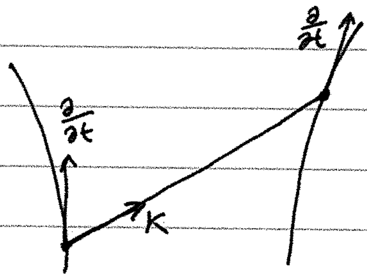
where $k = +1, 0,$ or -1.

2. (Redshifts)

Consider observers in the Friedman metric with velocities

$$U = \frac{\partial}{\partial t}.$$

Consider a null ray with tangent K^a between two such observers. Is K_t a constant of the motion?



Let $\omega = -K \cdot U$. If λ is an affine parameter on the null ray, show that

$$\frac{d\omega}{d\lambda} = -\dot{a}a K^i K^j \gamma_{ij},$$

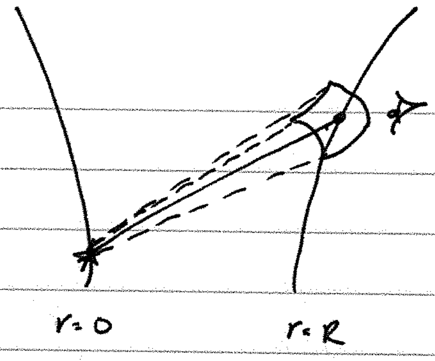
where $\dot{a} = \frac{da}{dt}$. If the ray is emitted at t_E and observed at t_0, show that

$$\frac{\omega_0}{\omega_E} = \frac{a(t_E)}{a(t_0)}.$$

What does this mean physically?

3. (Measured Flux)

Consider a star emitting light with power P. Without loss of generality, let the star be at $r=0$. We observe the star at $r=R$, $t=t_0$. What flux, F, do we measure?



n.b. For the sake of nomenclature,

- we usually normalise $a(t_0) = 1$ where t_0 is current time

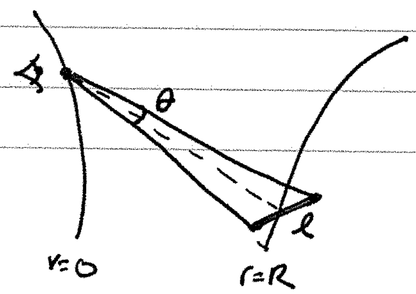
- we write $1 + z(t_E) = \frac{a(t_0)}{a(t_E)}$ for the 'redshift factor'

- the quantity

$$d_L = (1 + z(t_E))\, a(t_0) R,$$

where R is as in Q3, is called the 'luminosity distance.' Can you explain why it's called that?

4. (Measured Angles)
We observe a rod, length ℓ, which is oriented normally to our line of sight. The light leaves the rod at $t = t_E$. Suppose we are at $r = 0$ and the rod is at $r = R$. What angle does the rod subtend in our sky?



An object subtends a solid angle Ω in our sky and we recieve a flux F of light from it. What is the quantity

$$\Sigma = F\Omega$$

in physical terms? Find a formula for Σ in terms of

- the object's power P
- the proper area A of the object, normal to our line of sight
- the comoving distance R between us and the object.

Dynamics

5. (Einstein's Equation)

The Friedman metric has nonzero Christoffel symbols

$$\Gamma^0{}_{ij} = a\dot{a}\,\gamma_{ij}, \qquad \Gamma^i{}_{0j} = \frac{\dot{a}}{a}\delta_{ij}, \qquad \Gamma^i{}_{jk} = \breve{\Gamma}^i{}_{jk},$$

where $\breve{\Gamma}^i{}_{jk}$ are the Christoffel symbols of γ_{ij}. Show that

$$\boxed{\begin{aligned} R_{ij} &= (a\ddot{a} + 2\dot{a}^2 + 2k)\,\gamma_{ij} \\ R_{00} &= -3\,\frac{\ddot{a}}{a}. \end{aligned}}$$

n.b. the Ricci tensor of γ_{ij} is $\breve{R}_{ij} = 2k\,\gamma_{ij}$.

Consider Einstein's equation $G_{ab} = 8\pi G T_{ab}$ with an ideal fluid for T_{ab}. For the Friedman metric, show that this becomes

$$\boxed{\begin{aligned} \left(\frac{\dot a}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} \\ 2\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} &= -8\pi G P . \end{aligned}} \qquad (*)$$

What equation follows from energy conservation,

$$\nabla_a T^a{}_b = 0 ?$$

Show that it follows from $(*)$. Also deduce the equation

$$2\frac{\ddot a}{a} = -8\pi G\left(\frac{1}{3}\rho + P\right).$$