Notes on Hilbert schemes of points

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Precis

Consider n unlabelled points on the line $\mathbb{A}^1_{\mathbb{C}}$. The symmetric quotient $\mathbb{C}^n \xrightarrow{p} S^n\mathbb{C}$ is nonsingular for an amusing reason. The stalk at $x \in S^n\mathbb{C}$ is

$$\mathcal{O}_{S^n\mathbb{C},x} = \left(\mathcal{O}_{\mathbb{C}^n,p^{-1}(x)}\right)^{\mathfrak{S}_n} = \mathbb{C}\left\langle X_1, ..., X_n\right\rangle^{\mathfrak{S}_n},$$

i.e. the symmetric polynomials. A basis for the symmetric polynomials is given by the coefficients of t^i in the manifestly symmetric product $\prod_{i=1}^n (1 + x_i t)$.[1] This gives n basic symmetric polynomials—call them $e_1, ..., e_n$—and so

$$\mathbb{C}\left\langle X_1, ..., X_n\right\rangle^{\mathfrak{S}_n} = \mathbb{C}\left\langle e_1, ..., e_n\right\rangle.$$

This is a fundamental result about symmetric polynomials. It shows that $S^n\mathbb{C}$ is nonsingular and of dimension n. Other basic observations are that $S^n\mathbb{C}$ is connected and it has trivial cohomology since $H^*(S^n\mathbb{C}) = H^*(\mathbb{C})^{\mathfrak{S}_n}$. Moreover, it is stratified by the partitions of n. For instance, the stratum corresponding to (1^n) is $\mathbb{C}^n \backslash \triangle$, where $\triangle$ is the 'big' diagonal containing all collisions.[2] The symmetric product $S^n\Sigma$, for any nonsingular curve Σ, is nonsingular for the same reason that we described above. It parameterises n-tuples of points in Σ counted with multiplicity. It is an example of a Hilbert scheme of points. This essay concerns the analogous situation for points on surfaces. For a surface X, the Hilbert scheme of points, denoted $X^{[n]}$, is no longer a symmetric product. However, it is nonsingular and irreducible: properties that do not hold for dimensions greater than two. The cohomology of $S^n\Sigma$ was computed by MacDonald. If Σ has Betti numbers b_i, the Poincaré polynomials of the $S^n\Sigma$ are generated by[3]

$$\sum_{i,n=0}^{\infty} (-1)^i x^i t^n \dim H^i(S^n\Sigma) = \frac{(1 + xt)^{b_1}}{(1 - t)^{b_0}(1 - x^2 t)^{b_2}}. \tag{1}$$

A similar formula exists for the Poincaré polynomials of $X^{[n]}$ due to Göttsche. The direct sum of the cohomology rings of $X^{[n]}$ for all n form a Hopf algebra. This is discussed in section 3.2. We begin in section 1 by recalling the general construction of Hilbert schemes parameterising subschemes. To illustrate the main ideas in a concrete setting we first consider points on the

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[1] See Equation (2.2) of MacDonald's book, and the surrounding discussion. [1]

[2] The cohomology of $\mathbb{C}^n \backslash \triangle$ can be computed using a Serre-Leray spectral sequence. We have a fibration $F \hookrightarrow E \to B$, where $B = \mathbb{C}^{n-1} \backslash \triangle$ and $E = \mathbb{C}^n \backslash \triangle$. For a choice of basepoint in B, the fibre F is $\mathbb{C}$ with the chosen $n-1$ points removed. F has the cohomology of a wedge product of circles. The pure braid group $\pi_1(B)$ has a trivial action on $H^*(F)$ because is does not permute the factors. So one finds $H^*(E) = H^*(F) \oplus H^*(B)$ and we can now deduce $H^*(E)$ by an inductive argument: it is the cohomology of

$$\bigoplus_{i=1}^{n-1} \vee^i S^1.$$

For this calculation, see Arnold *The cohomology ring of the colored braids group*, 1968. [2] In quantum field theory, some amplitudes can be expressed as integrals over $\mathbb{C}^n \backslash \triangle$, and identities relating to the pure braid group are implicitly used. (M. Kapranow, unpublished)

[3] This is adapted from MacDonald's paper *The Poincare Polynomial of a Symmetric Product*, 1962. [3]

affine plane in section 2. We give an explicit presentation of the Hilbert scheme, show that it is non-singular and admits a symplectic form. In section 3, we show the corresponding results for a general surface. This includes Mukai's construction of a symplectic form on the Hilbert scheme.

Contents

1 Hilbert schemes

Given a scheme X, we are interested in studying its subschemes. For any such subscheme, Z, we have an associated ideal sheaf, $\mathcal{I}_Z$, that roughly corresponds to the 'vanishing ideal' of classical geometry. To be precise, an ideal sheaf $\mathcal{I}$ is a subsheaf of $\mathcal{O}_X$. Given such an $\mathcal{I}$ we can consider the support of $\mathcal{O}_X/\mathcal{I}$. Recall that the support of a sheaf $\mathcal{F}$ is the set of points for which the stalk of $\mathcal{F}$ is non-zero. Equivalently, $\mathrm{Supp}(\mathcal{F})$ is the union of the supports of all sections $s \in \mathcal{F}(U)$—where the support of s is the set of points where the image of s in the stalk is nonzero. This second definition allows us to write

$$\mathrm{Supp}(\mathcal{F})^c = \bigcap \mathrm{Supp}(s)^c.$$

$\mathrm{Supp}(s)^c$ is open in the Zariski topology: if $s_x = 0$, then, by definition, $s|_U = 0$ for some open U containing x. So we see that $Z = \mathrm{Supp}(\mathcal{O}_X/\mathcal{I})$ is closed. We can make Z into a scheme by defining $\mathcal{O}_Z = \iota^{-1}(\mathcal{O}_X/\mathcal{I})$, where $\iota : Z \to X$ is the inclusion. Going in the opposite direction, for any closed subset Z we can define a presheaf

$$\mathcal{I}_Z(U) \equiv \{f \in \mathcal{O}_X(U) \mid f(x) = 0, x \in U \cap Z\}.$$

This is an ideal sheaf by construction.[4] As sets, we have $Z = \mathrm{Supp}(\mathcal{O}_X/\mathcal{I}_Z)$. However, they needn't be the same as schemes. A scheme $(Z, \mathcal{O}_Z)$ is reduced if the ideals associated to $\mathcal{I}$ are radical. We might start with a non-reduced subscheme $(Z, \mathcal{O})$. Z as a set defines a sheaf of radical ideals, $\mathcal{I}$ and the associated subscheme, $(\mathrm{Supp}(\mathcal{O}_X/\mathcal{I}) = Z, \iota^{-1}(\mathcal{O}_X/\mathcal{I}))$, will be reduced. In this way we construct a reduced scheme Z_{red} with the same topological support as Z. In sum, we have a correspondence between the reduced subschemes of X and the radical quasi-coherent ideals. We can think of these ideals as kernels of quotients of $\mathcal{O}_X$. In other words,

[4] It is also quasi-coherent. Let U be affine. We see that $f(x) = 0$ iff f is in the prime ideal $\mathfrak{p}_x$. So $\mathcal{I}(U) = \bigcap \mathfrak{p}$. By further refining to affine opens D_f we get a basis in which $\mathcal{I}$ is q-coh.

$\mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z$ is exact. In section 1.3 we will explain that this puts our moduli problem in the general context of describing families of quotients of a coherent sheaf. Before this, we must recall the idea of a flat family.

1.1 Flat families

In classical geometry, we might describe a parameterised family of projective varieties as a surjective morphism $X \to S$ whose fibres are 'similar'. We can make this idea precise in algebra using flatness. In A-mod, a module M is called flat if $\otimes_A M$ is exact (similarly for mod-A). By tensoring M with $0 \to I \to A \to A/I \to 0$ we see that flatness implies

$$IM \simeq I \otimes_A M$$

for any ideal I. Indeed, the converse also holds.[5] Now consider an injective morphism of rings, $\pi : A \to B$, and the corresponding morphism of affine schemes $X = \mathrm{Spec}(B) \to S = \mathrm{Spec}(A)$. Let M be the ring B regarded as an A-module via π. For maximal ideals $\mathfrak{m} \subset A$ we get subvarieties $X_\mathfrak{m} = \mathrm{Spec}(\pi(\mathfrak{m}))$ of X. Suppose M is flat. Then the restriction of the function ring to $X_\mathfrak{m}$, regarded as an A-module, is

$$M/\mathfrak{m}M \simeq M \otimes_A A/\mathfrak{m}.$$

If S is affine space over some field k, then $A/\mathfrak{m} = k$ for all maximal ideals. Given this, every member $X_\mathfrak{m}$ of the family parameterised by S has the same coordinate ring. Such an M, then, is our prototypical 'flat family'.

Example. We could take $A = k[x]$ and $B = k[x, y]/(y^2 - x)$. Then A has only one nontrivial ideal, (x), and B satisfies the condition to be flat as an A-module.

Now we describe the generalisation to schemes. The natural definition is to call $\mathcal{F}$ a flat $\mathcal{O}_X$-module if $\otimes \mathcal{F}$ is exact in the category of $\mathcal{O}_X$ modules.[6] This definition clearly agrees with our definition for ordinary modules. Moreover, since taking stalks of $\mathcal{O}_X$ modules respects tensor products, we see that the stalks $\mathcal{F}_x$ are flat as $\mathcal{O}_{X,x}$ modules.[7] The converse is also true, and 'flatness at every point' can be taken as the definition of flatness. It is a fact from commutative algebra that a finitely generated module over a local (noetherian) ring is flat iff it is free. It follows that a coherent sheaf $\mathcal{F}$ on a noetherian scheme is flat iff it is locally free, since both properties—flatness and being locally free—can be checked on affine opens where we are reduced to ordinary modules. Given a morphism of schemes $Z \to S$, we can regard the structure sheaf $\mathcal{O}_Z$ as an $\mathcal{O}_S$ module. Just as in the affine case, we call Z a flat family if $\mathcal{O}_Z$ is flat as an $\mathcal{O}_S$ module.

1.2 Hilbert polynomials

Consider next a flat family of projective schemes (viewed as subschemes of $\mathbb{P}^n$) over $\mathrm{Spec}(A)$ for A local and noetherian. Equivalently, by our opening discussion, we can consider a coherent sheaf F on $\mathrm{Spec}(A) \times \mathbb{P}^n$, flat over $\mathrm{Spec}(A)$. Restricting this to open affines we have $F|_U = \widetilde{M}$, for some flat A-module M. This then presents $H^0(X, F)$ (or its twisted counterparts) as a flat A-module, since we see that the Cech resolution of $H^0(X, F)$ is a resolution of flat A-modules.

[5] See Proposition 3.2.4 of Weibel. [4]

[6] We have a tensor product of $\mathcal{O}_X$ modules by sheafifying the tensor product of modules.

[7] For the stalks of a tensor product see Stacks Lemma 17.15.1.

Now, it is essentially a homological algebra fact (which we will prove in a restricted sense) that the function

$$s \mapsto \sum (-1)^i \dim_{\kappa(s)} H^i(X_s, F_s)$$

is locally constant on S. Here $\kappa(s) = \mathcal{O}_{S,s}/\mathfrak{m}_s$ is the residue field. If we replace F by $F(m)$, we can ignore $i > 0$ for sufficiently large m by Serre's vanishing theorem. So consider

$$P_{F_s}(m) = \dim_{\kappa(s)} H^0(X_s, F_s(m)),$$

the Hilbert polynomial of F_s. Since A is local Noetherian, the kernel of any surjection $A \to k$ is finitely generated and we can choose a presentation $A^q \to A \to k \to 0$. Moreover, we let s be the closed point. Then $\kappa(s) = k$ and, tensoring with $H^0(X, F(m))$, we find

$$H^0(X, F(m))^q \to H^0(X, F(m)) \to H^0(X, F(m)) \otimes \kappa(s) \to 0. \tag{2}$$

Tensoring with F, we can regard $F \otimes_A \kappa(s) = F_s$ as the fibre over s. Then have an exact sequence $F^q \to F \to F_s \to 0$. Taking global sections and comparing to (2) we find

$$H^0(X_s, F_s(m)) \simeq H^0(X, F(m)) \otimes_A \kappa(s).$$

This means that

$$\dim_{\kappa(s)} H^0(X_s, F_s(m)) = \dim_A H^0(X, F(m)).$$

So $P_{F_s}(m)$ is locally constant with respect to s. This realises our original intuition that the subschemes in a flat family are geometrically similar. We do not lose anything by assuming s is a closed point since taking cohomology commutes with flat base change.[8] The converse statement also holds.[9]

Theorem 1. *Let A be local noetherian. Then $\mathcal{F}$ is flat over Spec(A) iff the Hilbert polynomials of $\mathcal{F}_t$ are constant in t.*

1.3 Quot functors

Since our aim is to study subschemes of a projective scheme, X, we define

$$\mathfrak{H}_X(T) = \{Z \subset X \times S \mid Z \text{ is flat over } T\}.$$

This is a (contravariant) functor from schemes to sets, and a morphism $f : T \to T'$ induces a map $X \times T' \to X \times T$ which sends $Z \mapsto (1 \times f)^{-1}(Z)$. Theorem 1 shows that the Hilbert polynomial defines a stratification of $\mathfrak{H}_X$. For each polynomial p we can define subfunctors $\mathfrak{H}_X^p$ which associate to T the flat families over T whose fibres have Hilbert polynomial p. We are interested in studying the case of points and we set $p = n$ for n points. In good circumstances (X is Noetherian), the functors $\mathfrak{H}_X^n$ are represented by schemes which we denote $X^{[n]}$. This means that there is a natural transformation from $\mathfrak{H}_X^n$ to $\text{Hom}(-, X^{[n]})$. Given $X^{[n]}$, we construct a natural transformation by choosing a family $\mathcal{E}$ in $\mathfrak{H}_X^n(X^{[n]})$. Then, for every morphism $\phi : T \to \text{Hilb}_X$, the corresponding flat family over T is given by the pull back of $\mathcal{E}$ by ϕ. (This is Yoneda's construction. $\mathcal{E}$ is called the universal family.) Finally, a flat family of subschemes over T is equivalent to a quotient sheaf $\mathcal{O}_X \xrightarrow{q} \mathcal{O}_Z$ such that $\mathcal{O}_Z$ is flat over T. (We take it to be understood that we are considering the pull backs of $\mathcal{O}_X$,$\mathcal{O}_Z$ to $X \times S$ via the projection $X \times S \to X$.) The kernel of q is $\mathcal{I}_Z$. So, recalling our opening discussion, two quotients q and q'

[8]See Proposition 9.3 in Hartshorne. [5]
[9]See Theorem 9.9 in Hartshorne. [5]

give the same family if $\ker(q) = \ker(q')$. This suggests a generalisation of $\mathfrak{H}_X$. For any coherent sheaf $\mathcal{G}$ on X, we could take

$$\mathfrak{Q}_{\mathcal{G},X}(T) = \{\text{quotients } \mathcal{G} \to \mathcal{F} \mid \mathcal{F} \text{ is flat over } T\}/\sim .$$

We regard two quotients q and q' as equivalent under $\sim$ if $\ker(q) = \ker(q')$.[10] Once again, it is understood that $\mathcal{G}$ and $\mathcal{F}$ are pulled back to $X \times T$. We recover $\mathfrak{H}_X$ as the functor $\mathfrak{Q}_{\mathcal{O}_X,X}$. The Hilbert polynomial $\chi(\mathcal{G} \otimes L^m)$, for some line bundle L, gives us a stratification of $\mathfrak{Q}_{\mathcal{G},X}$ into subfunctors $\mathfrak{Q}^{p,L}_{\mathcal{G},X}$. For $\mathcal{G} = \mathcal{O}_X$ and $L = \mathcal{O}_X$ we recover $\mathfrak{H}^p_X$. When X is Noetherian and L is very ample, Grothendieck showed that $\mathfrak{Q}^{p,L}_{\mathcal{G},X}$ is representable.[11] (Grothendieck took X and T to be S-schemes and $\mathfrak{Q}_{\mathcal{G},X}$ as a functor from Sch_S to Sets. Putting $S = \mathrm{Spec}(k)$, we recover the functors introduced here.)

2 Example: points on the affine plane

The zero-dimensional sub-schemes of $X = \mathbb{A}^2$ are given by prime ideals $I \subset R = k[x_1, x_2]$ whose coordinate rings R/I have finite k-dimension n corresponding to the number of points.[12] We want to explicitly construct the Hilbert scheme parameterising these sub-schemes. It is clearly not the symmetric product $S^n\mathbb{A}^2$ because we must include ideals such as $I = \langle x_1^2, x_2 \rangle$ whose coordinate rings have 'more functions than points'.[13] Fixing n, we have for any I an isomorphism

$$\alpha : R/I \to V.$$

This is uniquely determined up to the $\mathrm{GL}(V)$ action. Under α, the actions of the x_i on R/I become two commuting matrices, M_i, on V. In this way, an ideal I gives the data of two commuting matrices, up to the action of $\mathrm{GL}(V)$. To reverse this construction, suppose we are given two commuting matrices. We would like to find a surjection $\beta : R \to V$ whose kernel is I. If we take $\beta : f \mapsto f(M_1, M_2)v$, for some vector v, then this inverts the construction of M_i from I since $\beta(x_i f) = M_i \beta(f_i)$. However, we must also demand that β is surjective. With this refinement in mind, we arrive at the following equivalence (of sets)

$$\{\text{prime ideals } I \subset R \mid \dim_k(R/I) = n\} \quad \longleftrightarrow \quad \mathcal{Q} = \mathcal{P}/GL(V), \tag{3}$$

where

$$\mathcal{P} = \{M_1, M_2, [v] \mid [M_1, M_2] = 0, \text{ and } \langle M_1, M_2 \rangle v = V\}.$$

We write $[v]$ to denote the class of v under multiplication by scalars. Nakajima calls the second condition the 'stability condition.'

Proposition 2. (Nakajima.) *$\mathcal{Q}$ is smooth of dimension $2n$.*[14]

Proof. The $GL(V)$ action on $\mathcal{P}$ is given by $\beta : GL(V) \to End(\mathcal{P})$ where

$$\beta(g) : (M_1, M_2, v) \mapsto (gM_1g^{-1}, gM_2g^{-1}, gv).$$

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[10] In other words, $(\mathcal{G} \xrightarrow{q} \mathcal{F}) \sim (\mathcal{G} \xrightarrow{q'} \mathcal{F}')$ if there exists $\mathcal{F} \to \mathcal{F}'$ compatible with q and q'.

[11] Théorème 3.2 in Séminaire Bourbaki, 1960/61, no. 221 (IV. Les schémas de Hilbert). [6] His construction uses an embedding into an appropriate Grassmannian. Modern presentations are given by Nitsure in Section 5.5 of *Fundamental Algebraic Geometry* [7] and by Huybrechts-Lehn in Chapter 2 of their book. [8]

[12] As a ring, R/I has dimension zero because any prime ideal $J \supset I$ would be a sub-variety of strictly smaller dimension.

[13] Recall the example in section 1 of $k[x]/x^2$ and k.

[14] See Theorem 1.14 in Nakajima's book. [9]

The stabiliser of this action is trivial.[15] So $\mathcal{Q}$ is smooth provided that $\mathcal{P}$ is smooth. To see that $\mathcal{P}$ is smooth we define the map $\alpha : (M_1, M_2, v) \mapsto [M_1, M_2]$ and check that $\mathrm{d}\alpha$ has constant rank whenever v satisfies the stability condition. At the point (M_1, M_2, v) we have

$$\mathrm{d}\alpha(N_1, N_2, w) = [M_1, N_2] + [N_1, M_2].$$

The cokernel is $\mathrm{coker}(\mathrm{d}\alpha) = \mathfrak{gl}_\mathfrak{n}/I$ where I is the ideal generated by ad_{M_1} and ad_{M_2}. When v satisfies the stability condition, the vectors $M_1^a M_2^b v$ form a basis for V. For $N \in \mathrm{coker}(\mathrm{d}\alpha)$, $(N+I)M_1^a M_2^b v = M_1^a M_2^b (N+I)v$. So N determines and is determined by $Nv \in V$. It follows that $\ker(\mathrm{d}\alpha)$ has constant dimension $n^2 + 2n$. So $\mathcal{Q}$ is smooth. To find the dimension we use that the Zariski tangent space to $\mathcal{Q}$ at a point is given by $\ker(\mathrm{d}\alpha)/\mathrm{im}(\mathrm{d}\beta)$ (Proposition 5). The derivative of β at the point (M_1, M_2, v) is explicitly

$$\mathrm{d}\beta(h) = ([h, M_1], [h, M_2], hv),$$

for h in the Lie algebra $\mathfrak{gl}(V)$. Since $\ker(\mathrm{d}\beta)$ is zero when v satisfies the stability relation, we have $\dim \mathrm{im}(\mathrm{d}\beta) = \dim \mathfrak{gl}(V) = n^2$. So we see that $\dim(\mathcal{Q}) = n^2 + 2n - n^2 = 2n$. □

Proposition 3. *$\mathcal{Q}$ represents the Hilbert functor.*[16]

Proof. It suffices to explicitly construct the universal family. (Recall section 1.3.) Indeed, let $\mathcal{E} \subset \mathcal{Q} \times \mathbb{A}^2$ be the graph which associates every $[Z] \in \mathcal{Q}$ the corresponding subscheme of $\mathbb{A}^2$. This is a flat family with respect to the first projection because, by the construction of $\mathcal{Q}$, every fibre has the same Hilbert polynomial. If $Z \subset \mathbb{A}^2 \times S$ is a family of subschemes with Hilbert polynomial n, we want to find $\phi : S \to \mathcal{Q}$ such that $\mathcal{E}$ pulls back to give Z. Let π be the projection to S. Then, taking S to be affine open, $\pi_* \mathcal{O}_Z$ is locally free of rank n. Moreover, putting $\mathbb{A}^2 = \mathrm{Spec}(k[z_1, z_2])$, we see that multiplication by z_i induces endomorphisms M_i of $\pi_* \mathcal{O}_Z$ and the section $1 \in \mathcal{O}_Z$ gives some $v \in \pi_* \mathcal{O}_Z$. In sum, we have a map $\phi : S \to \mathcal{Q}$. By construction, the pull back of $\mathcal{E}$ by ϕ gives a family with the same (M_i, v) as Z (up to $\mathrm{GL}(V)$). By our remarks preceding equation (3) this means that the pull back is Z. □

2.1 Tangent spaces

In this section we prove Proposition 5 which was needed in Proposition 2. The tangent space at $[Z] \in \mathcal{Q}$ is the linear space of morphisms $\mathrm{Spec}(k(\epsilon)) \to \mathcal{Q}$ such that the image of $\mathrm{Spec}(k) \to \mathrm{Spec}(k(\epsilon)) \to \mathcal{Q}$ is Z. When $\mathcal{Q}$ represents the Hilbert functor, such morphisms are in correspondence with flat families of zero-dimensional schemes over $\mathrm{Spec}(k(\epsilon))$ such that the fibre over the point $\mathrm{Spec}(k) \to \mathrm{Spec}(k(\epsilon))$ is the scheme Z. If Z is the affine scheme $\mathrm{Spec}(R/I)$, we claim that these flat families in correspondence with homomorphsims $I \to R/I$ and we arrive at

Proposition 4. $T_I \mathcal{Q} = \mathrm{Hom}_R(I, R/I)$.

Proof. Recall that M is flat over A iff $I \otimes_A M = IM$ for all ideals I. We take $A = k(\epsilon)$, whose only ideal is (ϵ). Then M is flat over A iff $(\epsilon) \otimes_{k(\epsilon)} M = (\epsilon)M$. In our particular case, M is a module over $k(\epsilon)$ of the form $R \otimes_k k(\epsilon)/J$ for some J. The fibre over $\epsilon = 0$ is R/I, and so $J = I$ modulo ϵR. The flatness condition on M then reads $(\epsilon) \otimes_{k(\epsilon)} M = \epsilon M$. But $\epsilon M = \epsilon(R/I)$. So flatness requires that $\epsilon R \cap J = \epsilon I$. Any such J determines an R-homomorpshim $I \to R/I$ and vice versa. □

[15] If g is in the stabiliser of (M_1, M_2, v), then $gv = v$ so that $ker(g-1) \supset v$. But $gM_i = M_i g$, and so $ker(g-1)$ is an invariant subspace under the action of $\langle M_1, M_2 \rangle$. The 'stability condition' implies that v cannot be contained in a proper invariant subspace of $\langle M_1, M_2 \rangle$. So $\mathrm{Stab}(M_1, M_2, v) = \{1\}$.

[16] See Theorem 2.2.1 in de Cataldo & Migliorini *The Douady space of a complex surface*, 1998. [10]

Now take the presentation

$$0 \to I \to R \to R/I \to 0$$

and consider the associated exact sequence

$$\mathrm{Hom}_R(R/I, R/I) \to \mathrm{Hom}_R(R, R/I) \to T_I\mathcal{Q} \to \mathrm{Ext}^1(R/I, R/I) \to \mathrm{Ext}^1(R, R/I) \to ...$$

Since R is free as an R-module, and since the first arrow is an isomorphism, we find

$$T_I\mathcal{Q} \simeq \mathrm{Ext}^1(R/I, R/I).$$

We can generalise this discussion to schemes by replacing modules and ideals by $\mathcal{O}_X$ modules and ideal sheaves. We obtain two similar long exact sequences, now of sheaves. The sheaf $\mathscr{H}om(F, G)$ is formed from the pre-sheaf $\mathrm{Hom}_{\mathcal{O}_X|_U}(F|_U, G|_U)$ and the right derived functors of $\mathscr{H}om(F, -)$ are called $\mathscr{E}xt^i(F, -)$. Since $\mathscr{H}om(\mathcal{O}_X, -)$ is the identity, $\mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{I}_Z) = 0$ and the long exact sequence gives an isomorphism of sheaves $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z) = \mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{I}_Z)$. Generalising, we define $T_Z\mathcal{Q} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z)$ and consider the exact sequence, now of groups,

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_Z) \to T_Z\mathcal{Q} \to \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_Z) \to ...$$

Since $\mathrm{Hom}(\mathcal{O}_X, -)$ is the global sections functor, we see that $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_Z) = H^1(X, \mathcal{O}_Z)$. So

$$T_Z\mathcal{Q} \subset \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z). \tag{4}$$

We will use this in Theorem 10, Section 3.

Proposition 5. *The Zariski tangent space to $\mathcal{Q}$ at a point is given by* $\ker(d\alpha)/\mathrm{im}(d\beta)$, *where these maps are those given in Proposition 2.*

Proof. We use the identification of the tangent space with $\mathrm{Ext}^1(R/I, R/I)$. Our task is to compute the middle homology of the total complex of $\mathrm{Hom}_R(P, S)$ formed from resolutions P and S of R/I. Consider $0 \to V \to V \oplus V \to V \to 0$ with the first arrow $v \mapsto (vM_1, vM_2)$ and the second arrow $(v, w) \mapsto vM_2 - wM_1$. This is exact because M_1 and M_2 commute. Tensoring with the analogous complex formed by left-multiplication, we get a total complex of vector spaces

$$\mathfrak{gl}_n \to \mathfrak{gl}_n \oplus \mathfrak{gl}_n \to \mathfrak{gl}_n$$

with maps $X \mapsto ([X, M_1], [X, M_2])$ and $(X, Y) \mapsto [X, M_2] - [Y, M_1]$. The middle homology of this complex gives $\mathrm{Ext}^1(R/I, R/I) = \ker(\mathrm{d}\alpha)/\mathrm{im}(\mathrm{d}\beta)$. □

2.2 Induced symplectic structure

Our description of $\mathcal{Q}$ makes it clear that it possesses a symplectic form. Indeed, $\mathcal{P}$ is a complex vector space, $\mathrm{Hom}(V, V) \oplus \mathrm{Hom}(V, V) \oplus V$, which we can endow with a Hermitean inner product or, equivalently, a symplectic form. The map $\alpha(M_1, M_2) = [M_1, M_2]$ becomes a moment map for the $\mathrm{GL}(V)$ action. The moment map for the unitary group action is $[M_1, M_1^\dagger] + [M_2, M_2^\dagger] + v \cdot v^\dagger$. Together, these form a hyper-Kahler moment map $(\alpha, \mu_{\mathbb{C}})$ and exhibit $\mathcal{Q}$ as a hyper-Kahler quotient. In particular, $\mathcal{Q}$ is itself hyper-Kahler.[17] We do not pursue this geometric approach here. We will instead introduce the Hilbert-Chow morphism which will be used again in section 3. To every subscheme Z with Hilbert polynomial n we can associate the 0-cycle

$$\rho(Z) = \sum_{x \in \mathrm{Supp}(Z)} \mathrm{length}(\mathcal{O}_{Z,x})[x],$$

[17] This is carried out very explicitly in Chapter 3 of Nakajima's book. [9]

and this defines the Hilbert-Chow morphism, $\rho : \mathcal{Q} \to S^nX$, to the symmetric product. For smooth projective X, ρ is a surjective morphism and it is a desingularisation of $S^n\mathbb{C}^2$.[18] We will use this to construct a symplectic form on $\mathcal{Q}$ following Beauville. The partitions of n induce a disjoint decomposition of S^nX, and hence of $\mathcal{Q}$. For instance, consider the partitions (1^n) and $(1^{n-2}2)$. Let S^n_*X be the subset whose cycles form either one of these partitions. The pre-image $\mathcal{Q}_* = \rho^{-1}(S^n_*X)$, can be regarded as a blow-up of S^n_*X along the singular partition $(1^{n-2}2)$. Consider also the pre-image $X^n_* = \pi^{-1}(S^n_*X)$ of the symmetric quotient π—this is the set of n-tuples with at most two points the same. Then $X^n_* \cap \triangle$ is a codimension 2 subspace in X^n_*. (Notice that this is not the case for $X^n \cap \triangle$ in X^n.) We can arrive again at $\mathcal{Q}_*$ by blowing-up X^n_* along $X^n_* \cap \triangle$. Extending the symmetric action to $\text{Blow}_\triangle X^n_*$, we can take the symmetric quotient of $\text{Blow}_\triangle X^n_*$ to obtain $\mathcal{Q}$. Altogether, we have[19]

$$\begin{array}{ccc} \text{Blow}_\triangle X^n_* & \xrightarrow{\eta} & X^n_* \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ \mathcal{Q}_* & \xrightarrow{\rho} & S^n_*X \end{array}$$

Suppose X is symplectic with 2-form ω. On the product space $X^r = X \times ... \times X$ we obtain a 2-form $\psi = p_1^*\omega + ...p_r^*\omega$, where p_i denotes the ith projection. ψ is invariant under the symmetric action, and so is its pull-back $\eta^*\psi$ to $\text{Blow}_\triangle X^n_*$. It follows that $\mathcal{Q}_*$ inherits a holomorphic 2-form: i.e. there exists some φ such that $\eta^*\psi = \pi_1^*\varphi$. It remains to see that φ is a symplectic 2-form.

Proposition 6. *φ is non-degenerate on $\mathcal{Q}_*$.*

Proof. The pull-back $\eta^*\psi$ vanishes on the exceptional divisor, $\eta^{-1}(\triangle)$. So $|\eta^*\psi^n| = E$, where $|\cdot|$ denotes the divisor. However, $\eta^*\psi$ is invariant under the symmetric action, so we have $\eta^*\psi = \pi_1^*\varphi$ for some 2-form φ. But the projection π_1 is ramified along E so that $|\pi_1^*\varphi^n| = \pi_1^*|\varphi^n| + E$. Comparing we see that $|\varphi^n| = 0$, which shows that φ is a non-degenerate 2-form on $\mathcal{Q}_*$. □

Finally, if we work over $\mathbb{C}$, and $\dim\mathcal{Q} > 1$, the extension of φ to all of $\mathcal{Q}$ exists by Hartog's analytic extension theorem. Clearly these considerations also apply to our particular case, when $X = \mathbb{A}^2_{\mathbb{C}}$.

2.3 Cohomology

We now wish to describe the cohomology of $\mathcal{Q}$. We will do this by presenting an explicit cellular decomposition of $\mathcal{Q}$. First, let me make some observations about the strata $S^n_\alpha X$ of the symmetric product. Fix a partition of n, $\alpha = (1^{\alpha_1}...s^{\alpha_s})$, where $\sum_i i\alpha_i = n$. Points in the stratum $S^n_\alpha X$ correspond to choosing the locations of $\text{len}(\alpha) = \sum \alpha_i$ points. So $S^n_\alpha X$ has dimension $2\text{len}(\alpha)$. Therefore, the number of strata with codimension $2k$ is given by the number of ways to partition n into $n-k$ parts. We denote this number by $P(n, n-k)$. As we will see, the Betti numbers of $\mathcal{Q}$ are also given by $P(n, n-k)$.

Proposition 7. (Ellingsrud & Stromme) *The Betti numbers of $\mathcal{Q}$ are $b_{2k} = P(n, n-k)$, where $P(a, b)$ is the number of partitions of a into b parts.*[20]

[18] See Theorem 7.1.14 in *Foundational Algebraic Geometry.* [7]

[19] See Beauville's discussion preceding Proposition 5 in *Varietes Kahleriennes dont la premiere classe de chern est nulle*, 1988. [11]

[20] See Theorem 1.1 of their paper. [12]

In fact, Ellingsrud & Stromme established a cell decomposition of $\mathcal{Q}$ with one cell associated to every partition of n. We will now describe the key ideas in their derivation, omitting one calculation (for which we refer to Ellingsrud & Stromme). Our order of presentation follows Nakajima.[21]

Proof. (Of Proposition 7.) We will construct a perfect Morse function on $\mathcal{Q}$ using a torus action. Recall the description of $\mathcal{Q}$ in equation (3). A $\mathbb{T}^2$ action on $\mathcal{P}$ is

$$(M_1, M_2, v) \mapsto (t_1 M_1, t_2 M_2, t_1 v),$$

and this descends to $\mathcal{Q}$ under the $\mathrm{GL}(V)$ quotient. A moment map for this action is[22]

$$\mu : (M_1, M_2, v) \mapsto \begin{bmatrix} ||M_1||^2 + |v|^2 \\ ||M_2||^2 \end{bmatrix}.$$

Choosing some (α, β) in the Lie algebra $\mathfrak{t}$, we obtain a perfect Morse function $(\alpha, \beta) \cdot \mu$, for sufficiently generic (α, β). It greatly simplifies matters to choose β/α positive and arbitrarily close to 0. A fixed point of the action gives rise to a representation $\rho : \mathbb{T}^2 \to \mathrm{GL}(V)$. This is because, if $(M_1, M_2, v) \in \mathcal{P}$ descends to a fixed point in $\mathcal{Q}$, there must exist some $\rho(t) \in \mathrm{GL}(V)$ such that

$$(t_1 M_1, t_2 M_2, t_1 v) = (\rho(t) M_1 \rho(t)^{-1}, \rho(t) M_2 \rho(t)^{-1}, \rho(t) v)$$

for all $t \in \mathbb{T}^2$. The representation ρ induces a weight decomposition $V = \bigoplus_{k,l} V_{k,l}$ such that $\rho(V_{k,l}) = t_1^k t_2^l V_{k,l}$. Notice also that the weight of $V_{k,l}$ under our Morse function is $\alpha k + \beta l$. Since we have

$$t_1 M_1 \rho(t) = \rho(t) M_1,$$

we see that $M_1(V_{k,l}) \subset V_{k+1,l}$. Likewise, $M_2(V_{k,l}) \subset V_{k,l+1}$ and $v \in V_{1,0}$. By the stability condition, $\langle M_1, M_2 \rangle v = V$, this means that

$$V = \bigoplus_{k \geq 1,\, l \geq 0} V_{k,l},$$

and therefore

$$n = \sum_{k \geq 1,\, l \geq 0} \mathrm{dim} V_{k,l}.$$

Clearly we must have $[v] = V_{1,0}$, else V would be empty. So $\mathrm{dim} V_{1,0} = 1$. Proceeding in this way, $\mathrm{dim} V_{k,l}$ is either 0 or 1. Now recall that M_1, M_2 are commuting matrices. This means that $M_1(V_{k-1,l}) = 0$ iff $M_2(V_{k,l-1}) = 0$. Arrayed in a table, the nonzero $V_{k,l}$ thus form the shape of a Young tableaux for n boxes. In other words, the fixed points of our Morse function are in direct correspondence with the partitions of n. Fix a partition α and its conjugate β. To complete the proof we must show that the partition α corresponds to a cell of codimension $\mathrm{len}(\alpha)$. The lemma we need is[23]

Lemma 8. *At the given fixed point, the tangent space T has the weight space decomposition*[24]

$$T = \sum_{(i,j) s.t. dim V_{i,j} \neq 0} V_{\alpha_i - j + 1, -(\beta_j - i)} + V_{-(\alpha_i - j), \beta_j - i + 1}.$$

[21] See the Appendix to his paper *Heisenberg Algebra...*, 1995. [13]

[22] $||\cdot||$ is a Hermitian norm on $\mathrm{Hom}(V, V)$ (say, the Frobenius norm, $||A||^2 = \sum |A_{ij}|^2$.) and $|v|^2$ is a Hermitian norm on V. These come from a Hermitian metric on $\mathcal{P}$ which determines a symplectic form. With respect to this metric, μ is then clearly the moment map we want.

[23] This is adapted from §3 of Ellingsrud & Stromme. [12] We have adopted the notation and order of presentation used in the appendix to Nakajima's paper *Heisenberg Algebra...*, 1995. [13]

[24] Originally given as Lemma 3.2 in Ellingsrud & Stromme. [12] It is also proved as Proposition 5.7 in Nakajima's book. [9] The calculation proceeds using a presentation of $\mathrm{Ext}^1(R/I, R/I)$: an equivariant adaptation of the complex used in Proposition 5.

Note that each i,j with $\dim V_{i,j} \neq 0$ is a box in the tableaux associated to α. The only summands which have negative weight with respect to $(\alpha,\beta)\cdot\mu$ are the $V_{-(\alpha_i-j),\beta_j-i+1}$ whenever $\alpha_i - j > 0$.[25] The number of boxes for which $\alpha_i - j = 0$ is $\text{len}(\alpha)$. So the index of the critical point associated to α is

$$\text{ind} = \dim T_- = 2(n - \text{len}(\alpha)).$$

This is the dimension of the cell associated to α. Since the Morse function is perfect (it comes from a torus action), the cells are all inequivalent in homology. In particular, the Betti number b_{2k} is the number of cells with dimension $2k$, which is given by $P(n, n-k)$. □

Using this result we obtain a generating function for the Poincaré polynomials[26]

$$\sum_{i,n=0} t^{2i} q^n \dim H^{2i}(X^{[n]}) = \sum_{i,n=0} t^{2i} q^n P(n, n-i) = \prod_{m=1} (1 - t^{2(m-1)} q^m)^{-1}.$$

A particularly attractive corollary is that

$$\sum_{n=0} q^n \chi(X^{[n]}) = \prod_{m=1} \frac{1}{1-q^m}, \tag{5}$$

which resembles the Dedekind eta function.

2.4 Heisenberg algebra

Equation (5) is the character of an irreducible representation of the Heisenberg Lie algebra. This is no coincidence, as we will see in this section. Let V be a vector space. The Fock space modelled on V is the free commutative algebra $\mathcal{F}$ generated by $V_t = V \otimes_k tk[t]$. We will suppress $\otimes_k$ and write $v \otimes_k t^n = vt^n$ for $v \in V$. So we have

$$\mathcal{F} = \bigoplus_{i=0} \text{Sym}^i(V_t).$$

This is graded according to degree in t and we arrive at a decomposition

$$\mathcal{F} = \bigoplus_{n=0} \mathcal{F}^n.$$

A typical summand of $\mathcal{F}^n$ has the form

$$S_\alpha = \left(\text{Sym}^{\alpha(1)}(V) \otimes \text{Sym}^{\alpha(2)}(V) \otimes ...\right) t^n,$$

where $\sum_{m=1} m\alpha(m) = n$. That is, α is a partition of n. Each such summand has k-dimension $\dim(V)^n$. So

$$\dim_k \mathcal{F}^n = \dim(V)^n P(n),$$

where $P(n)$ is the number of partitions. The generating function is

$$\sum_{n=0} q^n \dim_k \mathcal{F}^n = \prod_{n=1} \frac{1}{(1-q^m)^{\dim(V)}}.$$

[25] In deducing this, we use that $\alpha_i - j \geq 0$ and $\beta_j - ij \geq 0$, as can be verified by, say, drawing the Young tableaux.

[26] To see the second equality, notice that the summands in the expansion of the product are in direct correspondence with ordered tuples $\{m_1, ...\}$ (not necessarily distinct). If $\sum_i i m_i = n$, the associated term is

$$t^{2(n-s)} q^n,$$

where s is the number of nonzero m_i. There are $P(n,s)$ such terms.

So we recover equation (5) for $\dim(V) = 1$. Indeed, we can identify $\mathcal{F}$ directly with the direct sum

$$\mathbb{H} = \bigoplus_n H^\bullet\left(\left(\mathbb{A}^2_{\mathbb{C}}\right)^{[n]}\right).$$

In the previous section, we found an explicit basis of cycles for $\mathbb{H}$ using a perfect Morse function. For fixed length n, we had one cycle C_α for every partition $\alpha \in \mathcal{P}_n$. Likewise, the summands S_α of $\mathcal{F}^n$ are in correspondence with the partitions of n. So we may identify $\mathbb{H}$ and $\mathcal{F}$ as vector spaces.[27] Since $\mathcal{F}$ is an algebra, we can regard $\mathbb{H}$ as an algebra. Let $\alpha \in \mathcal{P}_n$ and $\beta \in \mathcal{P}_m$ be partitions. Then the product in $\mathcal{F}$ of S_α and S_β is $S_{(\alpha,\beta)}$ where (α, β) is now a partition of $n + m$. In cohomology we may thus define a product

$$\mathbb{H}^n \otimes \mathbb{H}^m \to \mathbb{H}^{n+m} : [C_\alpha] \otimes [C_\beta] \mapsto [C_{(\alpha,\beta)}].$$

In fact, $\mathcal{F}$ is also an irreducible module for the infinite Heisenberg Lie algebra, $\mathcal{H}$. We can model $\mathcal{H}$ on V and take it to be the free commutative algebra on $V \otimes_k k[t, t^{-1}]$. An inner product $(\ ,\ )$ on V then induces a Lie bracket on $\mathcal{H}$ given by

$$[vt^a, wt^b] = \delta_{a+b}(v, w)1,$$

where 1 is the unit in k. $\mathcal{F}$ can be made into an $\mathcal{H}$-module in the following standard way. Fix $a > 0$. The action of vt^a on $\mathcal{F}$ is given by multiplication in $\mathcal{F}$. The action of vt^{-a} is defined by

$$(vt^{-a})1 = 0,$$

since the action on other elements is then implied by the Lie bracket. Finally, let vt^0 have the trivial action on $\mathcal{F}$. It is known that this is an irreducible representation of $\mathcal{H}$.[28] We can thus regard $\mathbb{H}$ as an irreducible module for the Heisenberg algebra.[29] We now interpret this geometrically. In section 3.3 we show that we can describe $C_\alpha \subset X^{[n]}$ as comprised of varieties with the form $Z_1 \cup Z_2 \ldots \subset \mathbb{A}^2$ where the Z_i are all supported at distinct points and Z_i has length α_i. Then $C_{(1,\alpha)}$ is obtained from the subvarieties in C_α by considering in addition a new point distinct from the others. Likewise, the action of $[C_{(m)}]$ on the cycle $[C_\alpha]$ amounts to 'adding' a (fuzzy) point of length m. Everything said here generalises to the case of any smooth surface X, so we postpone a more detailed discussion to section 3.3.

3 Points on surfaces

Throughout this section, X will be an irreducible, nonsingular surface. Having studied the case $X = \mathbb{A}^2_{\mathbb{C}}$ in the previous section, we see that many of the results generalise naturally to general

[27] This idea was employed by Vafa & Witten in a heuristic computation of the cohomology of $[X^n/\mathfrak{S}_n]$ regarded as an orbifold. This is pages 56 to 59 of their paper *A Strong Coupling Test of S-duality*, 1994. [14] 'Orbifold' cohomology was partially defined by Chen and Ruan in 2000. [15] The idea was fully anticipated by M. Kontsevich in an unpublished letter in 1996, as reviewed by Abramovich. [16] Fantechi and Göttsche showed that the orbifold cohomology of $[X^n/\mathfrak{S}_n]$ gives the cohomology of $X^{[n]}$ in Section 3 of their paper *Orbifold cohomology for global quotients*, 2001. [17] Their construction has since been translated into geometrical terms, and we explain this in section 3.3.

[28] The idea is that $\mathcal{F}$ cannot have an $\mathcal{H}$-invariant subspace. This is because the $\mathcal{H}$-orbit of any $w \in \mathcal{F}$ contains 1 and, therefore, is all of $\mathcal{F}$. This is common knowledge among undergraduate physics students. A proof is given as Theorem 1 in Ottesen's book about the Heisenberg group. [18]

[29] In fact, the algebraic character of $\mathcal{F}$ is given by the weight decomposition as

$$\mathrm{ch}_q(\mathcal{F}) = \sum_\lambda q^\lambda \dim(\mathcal{F}_\lambda) = \prod_{n=1} \frac{1}{(1-q^m)^{\dim(V)}}.$$

We thus recover equation (5) a third time. For the characters of infinite Lie algebras see §9.3 of Kac's book. For the Heisenberg Lie algebra in particular see §2.8. [19]

X. We begin by showing that $X^{[n]}$ is connected, irreducible and nonsingular.

Theorem 9. *The Hilbert scheme $X^{[n]}$ is connected.*

Proof. Since $X^{[1]} = X$ is connected, we reason by induction. Consider the universal family $Y \subset X^{[n]} \times X$ and its associated ideal sheaf $\mathcal{I}_Y$ with respect to $\mathcal{O}_{X^{[n]}\times X}$. We consider the Quot-scheme $P = \text{Quot}^1(\mathcal{I}_Y)$ of 1-dimensional quotients. We will construct a fibration

$$X \times X^{[n]} \xleftarrow{\ p\ } P \xrightarrow{\ q\ } X^{[n+1]}.$$

The idea is that q will be a surjective morphism. It is clear that P is connected, since $X \times X^{[n]}$ is connected by hypothesis. So constructing q will show that $X^{[n+1]}$ is connected. The morphism p is the usual projection associated to Quot. For any $(x, Z) \in X \times X^{[n]}$, the fibre of p is

$$\{\text{surjections } \mathcal{I}_Z \to \kappa(x)\}/\sim,$$

where $\mathcal{I}_Z$ is the $\mathcal{O}_X$ ideal sheaf for the variety Z and $\kappa(x)$ is the residue field of x. For any such surjection, λ, we have

$$0 \to \mathcal{I} \to \mathcal{I}_Z \xrightarrow{\lambda} \kappa(x) \to 0,$$

where $\mathcal{I}$ is the kernel of λ. Let $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}_Z$ and $\mathcal{O}_\lambda = \mathcal{O}_X/\mathcal{I}$. Then

$$0 \to \kappa(x) \to \mathcal{O}_W \to \mathcal{O}_Z \to 0.$$

Now $\mathcal{O}_Z$ has length n since $Z \in X^{[n]}$ so $\mathcal{O}_\lambda$ has length $n+1$. In this way, we have a scheme $W \in X^{[n+1]}$ for every λ. Conversely, given W we can choose a point $x \in W$ and some $f \in \mathcal{O}_W$ vanishing at x. The ideal of f defines a subscheme Z and the cokernel of $\mathcal{I}_W \to \mathcal{I}_Z$ is $\kappa(x)$. This defines a surjection $\mathcal{I}_Z \xrightarrow{\lambda} \kappa(x)$ which is in the fibre of p over (x, Z). □

In general it is not the case that Hilbert schemes of points are irreducible and nonsingular. For a nonsingular projective variety M, if $\dim(M) > 2$, $M^{[n]}$ will typically be reducible and singular for sufficiently large n. This is demonstrated by Iarrobino's construction.[30] For simplicity, we explain his construction only for the affine spaces $\mathbb{A}^d$. Fix a degree k. Let J be an ideal generated by s monomials in degree k. Let $\mathfrak{m}$ be the maximal ideal at some choice of origin. Then, for any such J, consider the ideal

$$I_J = \mathfrak{m}^{k+1} + J.$$

It is easy to compute the dimension of the associated subvariety. The coordinate ring R/I_J contains all monomials with deg $\leq k$, except for those in J. So the dimension of the subvariety is

$$n_{d,k,s} = \dim R/I_J = -s + \sum_{i=0}^{k} \binom{i+d}{d}.$$

We obtain a family of subvarieties by varying J. This family has dimension

$$m_{d,k,s} = \dim \text{Grass}\left(s, \binom{k+d}{d}\right) = s\left(\binom{k+d}{d} - s\right).$$

We can regard the $\text{Spec}(R/I_J)$ as belonging to $\left(\mathbb{A}^d\right)^{[n]}$. The expected dimension of $\left(\mathbb{A}^d\right)^{[n]}$ is $dn_{d,k,s}$. So if we can show that

$$m_{d,k,s} - dn_{d,k,s} > 0$$

[30] See especially section 3 of his paper *Reducibility of the families of 0-dimensional schemes on a variety*, 1972. [20]

for some choice of k and s, $\left(\mathbb{A}^d\right)^{[n]}$ is reducible. Indeed, fixing a dimension $d > 2$ and some $s \geq d$ we have

$$m - dn = (s-d)\binom{k+d}{d} + d\sum_{i=0}^{k-1}\binom{i+d}{d} - s(s+d).$$

Clearly this can be made positive for sufficiently large k. For instance, taking $d = 4$ and $s = 4$ we can make it positive by choosing $k = 3$. Since $n_{4,3,4} = 21$, we conclude that $\left(\mathbb{A}^4\right)^{[21]}$ is reducible. For any $d > 2$, Iarrobino's construction gives some $n(d)$ such that $\left(\mathbb{A}^d\right)^{[n]}$ is reducible for all $n > n(d)$. This doesn't work for $d = 2$. Indeed, Serre duality gives the following result.

Theorem 10. *$X^{[n]}$ is irreducible and nonsingular of dimension $2n$.*[31]

Proof. The Hilbert-Chow morphism (section 2.2) gives $X^{[n]}_{(1^n)} \simeq S^n_{(1^n)}X$. The stratum $S^n_{(1^n)}X$ is nonsingular of dimension $2n$. So the desired result for $X^{[n]}$ will follow if we bound the dimension of the tangent space above by $2n$. Recall from section 2.1 that the tangent space at a subvariety Z is $T_Z X^{[n]} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z)$. We showed there, equation (4), that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z) \subset \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_Z)$. Using the existence of a locally free resolution of $\mathcal{O}_Z$, we have the homological algebra fact[32]

$$\chi(\mathcal{O}_Z) = \sum(-1)^i \dim \mathrm{Ext}^i(\mathcal{O}_Z, \mathcal{O}_Z) = 0.$$

Since Z has Hilbert polynomial n, $H^0(\mathcal{O}_Z) \simeq k^n$. Serre duality then gives

$$\mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \simeq H^0(\mathcal{O}_Z \otimes K)^*,$$

where K is the canonical bundle on the surface. So the algebra fact gives $\dim T_Z X^{[n]} \leq 2n$ for all Z; i.e. there are no singular points. □

3.1 Symplectic structure

We have already seen Beauville's construction of a symplectic form on $X^{[n]}$ induced by a symplectic form on X. Here we arrive again at this result, but more explicitly, by describing Mukai's construction.

Theorem 11. (Mukai) *If X has a (holomorphic) symplectic form $\omega \in H^0(K_X)$, $X^{[n]}$ admits a closed 2-form.*[33]

Proof. Given ω, Serre duality gives $\omega^\vee \in H^2(\mathcal{O}_X)^\vee$. To get a 2-form on $X^{[n]}$ it will suffice to choose some

$$\gamma \in H^2(\mathcal{O}_X) \otimes H^0(\Omega^2_{X^{[n]}}).$$

We can regard γ as being a class in $H^\bullet(X \times X^{[n]}, \Omega^\bullet)$ by the Künneth theorem. We let p, q be the projections of $X \times X^{[n]}$ so that

$$\Omega^2 = p^*\Omega^2_X \oplus q^*\Omega^2_{X^{[n]}}.$$

If γ is closed with respect to $1 \otimes \mathrm{d}_{X^{[n]}}$ then

$$\langle \omega, \gamma \rangle \in H^0(\Omega^2_{X^{[n]}})$$

[31]This is discussed following Example 4.5.10 in Huybrechts & Lehn. [8]

[32]This follows by taking a locally free resolution E_i of $\mathcal{O}_Z$ and using $\sum(-1)^i \mathrm{rank}(E_i) = 0$.

[33]This is Example 0.4 of Mukai's paper *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*, 1984. [21] It is reproduced also as Proposition 10.3.2 in Huybrechts & Lehn. [8]

is a closed 2-form on $X^{[n]}$. It remains to construct such a d-closed γ. Let $Y \subset X \times X^{[n]}$ be the universal family and $\mathcal{I}$ its ideal sheaf. We consider the Atiyah class

$$A(\mathcal{I}) \in \mathrm{Ext}^1(\mathcal{I}, \mathcal{I} \otimes \Omega^1).$$

If $\mathcal{I}$ is locally free, $A(\mathcal{I})$ is d-closed.[34] Given this, we also have that

$$\mathrm{tr}_{\mathcal{I}} A(\mathcal{I}) \circ A(\mathcal{I}) \in H^2(X \times X^{[n]}, \Omega^2_{X \times X^{[n]}})$$

is d-closed. We can take γ to be the $(2,0)$ component of $\mathrm{tr}_{\mathcal{I}}(A(\mathcal{I}) \circ A(\mathcal{I}))$ with respect to the Kunneth decomposition. This is d-closed and we are done. (If $\mathcal{I}$ is not locally free, we can take a locally free resolution. Extending the definition of the trace, we obtain from this resolution a d-closed γ in the same way.) □

We can now unpack this construction locally. Given the splitting $\Omega^2 = p^*\Omega^2_X \oplus q^*\Omega^2_{X^{[n]}}$, let A' be the second component of $A(\mathcal{I})$. Restricting to $Z \in X^{[n]}$, A' induces a map

$$\mathrm{Hom}_X(\mathcal{I}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z),$$

using Serre duality.[35] Recall our construction of γ from composition and trace. So, restricted to Z,

$$\langle \omega, \gamma \rangle : T_Z X^{[n]} \times T_Z X^{[n]} \xrightarrow{A'} \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \times \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \longrightarrow H^2(\mathcal{O}_X) \xrightarrow{\omega} k. \tag{6}$$

The middle arrow is composition and trace. In the following theorem, we use a variant of this presentation to show when $\langle \omega, \gamma \rangle$ is non-degenerate.

Theorem 12. *Let X be K3, then $\langle \omega, \gamma \rangle$ is non-degenerate.*

Proof. Fix $Z \in X^{[n]}$. For any bounded complex of coherent sheaves, Serre duality gives a perfect pairing

$$\mathrm{Ext}^1(C^\bullet, K_X) \otimes H^1(X, C^\bullet) \to k.$$

induced by the evaluation map and the identification $H^2(X, K_X) \simeq k$. Taking $C^\bullet$ to be a locally free resolution of $\mathcal{O}_Z$, Serre duality induces a perfect pairing

$$\mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z \otimes K_X) \otimes \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \to k.$$

Now ω (regarded as a map $\mathcal{O}_X \to \mathcal{K}_X$) gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \times \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) & \longrightarrow & H^2(\mathcal{O}_X) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z \otimes K_X) \otimes \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) & \longrightarrow & k \end{array}.$$

Adapting equation (6), we can present $\langle \omega, \gamma \rangle$ at a point as

$$\langle \omega, \gamma \rangle : T_Z X^{[n]} \times T_Z X^{[n]} \to \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z \otimes K_X) \otimes \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \to k.$$

The first arrow is induced by A' and ω. The second arrow is induced by the Serre pairing. Since the second arrow is a perfect pairing, $\langle \omega, \gamma \rangle$ will be non-degenerate if the first arrow is an isomorphism. This is certainly the case when X is K3, since $K_X \simeq \mathcal{O}_X$. □

[34] To explain this, recall the definition of $A(L)$ for a line bundle L. If f_{ij} are transition functions for L, the 1-forms $\mathrm{d}\log f_{ij}$ form a d-closed Čech cocycle for $A(L)$. So $A(L)$ is d-closed.

[35] As a sheaf statement, A' induces a map $\Omega^\vee_{X^{[n]}} \to \mathscr{E}xt^1_p(\mathcal{I}, \mathcal{I})$ where $\mathscr{E}xt^1_p(\mathcal{I}, -) = R^1 p_* \mathscr{H}om(\mathcal{I}, -)$; see 10.1 of [HL].

In fact, one can see that $\dim H^0(\Omega^2) \leq 1$ and so $\langle \omega, \gamma \rangle$ must span $H^2(M)$.[36]

3.2 Cohomology

Our purpose here is to briefly put the results of sections 2.3 and 2.4 into a more general context. We give references in place of proofs. There is a general result due to Göttsche and Soergel for the cohomology of the Hilbert scheme of points on a nonsingular, irreducible surface.[37] The formula is

$$H^{2k}(X^{[n]}) = \sum_{\nu} H^{2(\alpha_1+...+\alpha_s)-2(n-k)}(S^{\alpha_1}X \times ... \times S^{\alpha_s}X), \tag{7}$$

where α is the partition $(1^{\alpha_1}...s^{\alpha_s})$. In the example $X = \mathbb{A}^2_{\mathbb{C}}$, the only nonzero Betti number of $X = \mathbb{C}^2$ is $b_0 = 1$. It follows that S^kX also has trivial cohomology (with $b_0 = 1$). Using the general formula, we can compute

$$\dim H^{2k}(X^{[n]}) = \sum_{\nu} \dim H^{2(\alpha_1+...+\alpha_s)-2(n-k)}(S^{\alpha_1}X \times ...) = \mathrm{card}\{\nu | \alpha_1 + ... + \alpha_s = n - k\}.$$

We thus arrive a second time at the same result. Starting with equation (7) we can derive a generating function for the Poincaré polynomials of $X^{[n]}$. MacDonald's result, which we gave earlier in equation (1), can be written

$$\sum_{n=0}^{\infty} t^n p(S^nX, z) = \prod_{j=0}^{\infty} \left(1 - (-1)^j z^j t\right)^{(-1)^{j+1}b_j},$$

where $p(S^nX, z)$ is the Poincaré polynomial. Equation (7) gives

$$\sum_{n=0}^{\infty} t^n p(X^{[n]}) = \sum_{n=1}^{\infty} \sum_{\alpha \in \mathcal{P}_n} t^n z^{2(n-\mathrm{len}(\alpha))} \left[\sum_j (-1)^j z^j \dim H^j(S^{\alpha_1}X \times ...) \right].$$

So, using the functoriality of p, we have

$$\sum_{n=0}^{\infty} t^n p(X^{[n]}) = \sum_{n=1}^{\infty} \sum_{\alpha} \prod_{i} \left(t^i z^{2i-2}\right)^{\alpha_i} p(S^{\alpha_i}X, z).$$

Summing over all n we get

$$\sum_{n=0}^{\infty} t^n p(X^{[n]}, z) = \prod_{i=0}^{\infty} \sum_{a=0} (t^i z^{2i-2})^a p(S^aX, z).$$

Finally, using MacDonald's formula, we obtain the following result

$$\sum_{n=0}^{\infty} t^n p(X^{[n]}) = \prod_{i=0}^{\infty} \prod_{j=0}^{\infty} \left(1 - (-1)^j z^{2(i-1)+j} t^i\right)^{(-1)^{j+1}b_j}. \tag{8}$$

This was first obtained by Göttsche using the Weil conjectures.[38] Putting $b_i = 0$ for all $i > 0$ and $b_0 = 1$ we recover the generating function that we derived for $X = \mathbb{A}^2_{\mathbb{C}}$ in section 2.3.

[36] This is part of Theorem 6.2.4 in Huybrechts & Lehn. [8] The bound comes from computing

$$H^0(X^n, \Omega^2)^{S_n} \simeq \left(H^0(X, \Omega^2)^{\otimes n}\right)^{S_n} \simeq H^0(X, \Omega^2) = k.$$

[37] This is Theorem 1 of their paper. [22] Their result makes use of the stratification of S^nX and exploits the decomposition theorem of Beilinson-Berstein-Deligne.

[38] See his paper, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface*, 1990. [23]

3.3 Back to curves

We will now generalise our remarks in section 2.4 about the cohomology of the Hilbert schemes of points. As before, take

$$\mathbb{H} = \bigoplus_{n=0} H^\bullet(X^{[n]}).$$

Let $V = H^\bullet(X)$, which we regard as a $\mathbb{Z}_2$ graded vector space with respect to the cohomological degree. The generalisation of section 2.4 is that $\mathbb{H}$ is isomorphic as a graded vector space to the free graded-commutative algebra on V_t. This is a corollary, by the structure theorem for Hopf algebras, of the following much more interesting theorem.

Theorem 13. (Grojnowski) $\mathbb{H}$ *can be made into a Hopf algebra whose primitive elements are isomorphic to* V_t.[39]

Proof. Fix m and n. Let $\Lambda_{m,n}$ be the closure of the subset

$$\{(I_m, I_n, I_{m+n}) \,|\, 0 \to I_m \to I_{m+n} \to I_n \to 0 \text{ is exact}\} \subset X^{[m]} \times X^{[n]} \times X^{[m+n]}.$$

Members of this subset correspond to two non-intersecting subvarieties of X with length m and n. In cohomology we have

$$[\Lambda_{m,n}] \in \mathbb{H}^n \otimes \mathbb{H}^m \otimes \mathbb{H}^{n+m}.$$

Let π_1, π_2 be the projections

$$X^{[m]} \times X^{[n]} \xleftarrow{\pi_1} X^{[m]} \times X^{[n]} \times X^{[m+n]} \xrightarrow{\pi_2} X^{[m+n]}.$$

Then we can define a map

$$m : \mathbb{H}^n \otimes \mathbb{H}^m \to \mathbb{H}^{n+m} : a \otimes b \mapsto \pi_{2*}(\pi_1^*(a \otimes b) \cap [\Lambda_{m,n}]),$$

which extends to a multiplication on $\mathbb{H}$. Let $(\ , \ )$ be the intersection pairing on $\mathbb{H}$. That is, (a, b) is nonzero only if $a, b \in \mathbb{H}^n$ have the same degree, in which case

$$(a, b) = \int_{[X^{[n]}]} a \cup b.$$

This is extended to the tensor algebra by the rule $(a \otimes a', b \otimes b') = (a, b)(a', b')$. Given this, the adjoint to m is the co-multiplication

$$m^* : \mathbb{H}^{n+m} \to \mathbb{H}^n \otimes \mathbb{H}^m : c \mapsto \pi_{1*}(\pi_2^*(c) \cap [\Lambda_{m,n}]),$$

which extends to a comultiplication on $\mathbb{H}$. The unit of m is the generator 1 of $\mathbb{H}^0$. Also m respects the $\mathbb{Z}_2$ grading by cohomological degree,

$$m(a, b) = (-1)^{|a||b|} m(b, a).$$

So $(\mathbb{H}, m, m^*, 1)$ is a graded commutative Hopf algebra. It remains to show that its primitive elements are isomorphic as a graded vector space to V_t. Let $M_n \subset X \times X^{[n]}$ be the subset of pairs (p, Z), where the scheme Z is supported only at the point p. Consider the projection

$$X \times X^{[n]} \xrightarrow{\pi} X.$$

[39] His argument is sketched in Section 3 of *Instantons and Affine Algebras I*, 1995. [24]

Notice that $[M_n] \cap \pi^*[Z] \in \mathbb{H}^n$. So we have a map

$$\iota_n : V \to \mathbb{H}^n : [Z] \mapsto [M_n] \cap \pi^*[Z].$$

Represent the (Poincaré dual of the) class $[Z]$ by a subvariety $Z \subset X$. Then the class $[M_n] \cap \pi^*[Z]$ could be represented by the subvariety in $X^{[n]}$ corresponding to length n subvarieties supported at a point which lies in Z. Notice that $\iota_n([Z])$ is a primitive element under m^* since subschemes $n[p]$ cannot be decomposed as $a \cup b$ such that $a \cap b = 0$. On the other hand, every weight n primitive element must be of this form. So the map induces $V \simeq \mathrm{Prim}(\mathbb{H}^n)$. This means that $\mathrm{Prim}(\mathbb{H}) \simeq V \otimes_k tk[t]$. □

It is known that a Hopf algebra is generated as a graded commutative algebra by its primitive elements.[40] So we have

Corollary 14. *As algebras, $\mathbb{H} = \mathcal{F}(V_t)$, where $\mathcal{F}$ denotes the free graded commutative algebra.*

This is what we found in section 2.4 for the example $X = \mathbb{A}^2_{\mathbb{C}}$. To be more explicit, we can write $V = V_+ \oplus V_-$ for the $\mathbb{Z}_2$ grading on V and then

$$\mathcal{F}(V_t) = \bigoplus_{i=0} \mathrm{Sym}^i(V_t^+) \oplus \mathrm{Asym}^i(V_t^-).$$

A typical element can be written in the form

$$\ldots \cdot [b]_j \cdot [a]_i \cdot 1,$$

where $[a], [b], \ldots$ are classes in V and $i, j, \ldots$ are positive integers. We denote by $[a]_i \in \mathbb{H}^i$ the primitive element of weight i obtained from $[a] \in V$ via the isomorphism $V \simeq \mathrm{Prim}(\mathbb{H}^i)$. The dots represent the multiplication m. Suppose $a \subset X$ represents the class $[a]$. Unpacking the definitions we see that

$$[a]_i \cdot 1 \in \mathbb{H}^i$$

can be represented by a cycle in $X^{[i]}$ comprised of those subschemes $Z \subset X$ of length i that are supported at a point in a. For the example $X = \mathbb{A}^2$, we gave an explicit basis for $\mathbb{H}$ in section 2.4. What we previously called $[C_\alpha]$ could now be written as

$$\ldots \cdot [1]_{\alpha_3} \cdot [1]_{\alpha_2} \cdot [1]_{\alpha_1} \cdot 1.$$

This explains the geometric interpretation of $[C_\alpha]$ given in section 2.4. Finally, we have the following generalisation of section 2.4.

Corollary 15. *$\mathbb{H}$ is a module for the ($\mathbb{Z}_2$ graded) infinite Heisenberg Lie algebra.*

It suffices to explain the definitions. The graded Heisenberg Lie algebra $\mathcal{H}$ modelled on V is the free graded commutative algebra on $V \otimes_k k[t, t^{-1}]$ with the quasi-Lie bracket

$$[[a]_i, [b]_j] = \delta_{i+j}(a, b).$$

The commutator is graded so that

$$[[a]_i, [b]_j] = [a]_i[b]_j - (-1)^{|a||b|}[b]_j[a]_i.$$

[40] This is true over a field of characteristic zero, assuming that the graded components $\mathbb{H}^i$ are finite dimensional and that m^* is an algebra homomorphism. Our present situation meets these requirements. For a review of Hopf algebras see the IHES notes by P. Cartier and section 2.5 for the structure theorem. [25] See also Theorem 3C.4 in Hatcher's book. [26]

If a and b have opposite degree in $\mathbb{Z}_2$ then $(a,b)=0$. This is because X is two dimensional, and therefore $X^{[i]}$ is $2i$ dimensional (by Theorem 10). It follows that two cycles $a, b \in V$ have vanishing intersection pairing unless a, b have the same degree mod 2. The algebra $\mathcal{H}$ thus decomposes into two sub-algebras, $\mathcal{H}^+$ and $\mathcal{H}^-$ according to the $\mathbb{Z}_2$ grading. $\mathcal{H}^-$ is the infinite Clifford algebra while $\mathcal{H}^+$ is the 'ordinary' infinite Heisenberg algebra that we described previously. $\mathbb{H}^+ = \mathrm{Sym}(V_t^+)$ is (as we discussed in section 2.4) an irreducible module for $\mathcal{H}^+$, and $\mathbb{H}^- = \mathrm{Asym}(V_t^-)$ is an irreducible module for $\mathcal{H}^-$. We have already seen that $\mathrm{Sym}(V_t^+)$ has character

$$\mathrm{ch}_q \mathbb{H}^+ = \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^{\dim(V^+)}},$$

as an $\mathcal{H}^+$ module. On the other hand, $\mathrm{Asym}(V_t^-)$ has character[41]

$$\mathrm{ch}_q \mathbb{H}^- = \prod_{m=1}^{\infty} (1+q^m)^{\dim(V^-)}.$$

So the character of $\mathbb{H}$ as a representation of $\mathcal{H}$ is given by

$$\mathrm{ch}_q \mathbb{H} = \prod_{m=1}^{\infty} \frac{(1+q^m)^{\dim(V^-)}}{(1-q^m)^{\dim(V^+)}}.$$

A brief study of Göttsche's formula reveals (putting $z=1$ in equation (8)) that this is the generating function for the Euler characteristics of the $X^{[n]}$.

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