Locally Acyclic Cluster Algebras & Surfaces

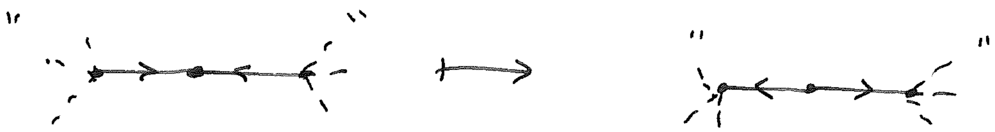
Given a quiver Q on $m+n$ vertices we define a subalgebra of

$$\mathbb{Q}(x_1, \ldots, x_{m+n}).$$

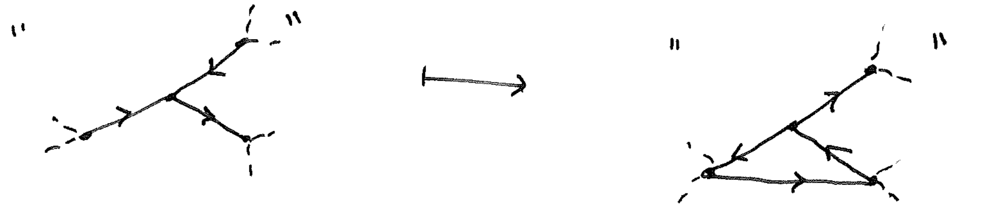
Let $Q_{mut} \subset Q$ be the subquiver labelled by $1 \ldots m$.

Each vertex of Q is associated to a variable x_i.

The quiver mutates according to the rules:



and



The variable associated to the mutated vertex is given by

$$x' = \frac{\pi_x^+ + \pi_x^-}{x},$$

where $\pi_x^+ = \prod_{\substack{\text{incoming}\\\text{vertices}}} x_i \qquad \pi_x^- = \prod_{\text{outgoing}} x_i.$

The subalgebra of $\mathbb{Q}(x_1 \cdots x_{m+n})$ defined by the polynomial algebra of all cluster variables obtained in this way is called $\mathcal{A}(Q)$.

The complex obtained by mutating Q is the cluster complex.

<u>Hard Thm</u> That complex is finite iff Q is Dynkin.

Given Q and variables $\underline{x} = (x_1, \dots, x_m)$, let Q_i be the quiver and $\underline{x}_i$ the variables obtained by mutating the x_i vertex.

e.g. The quiver $\bullet \overset{}{\underset{n}{\rightrightarrows}} \bullet = Q$ does not give a finite complex. One has:

$$\cdots\ \bullet\xrightarrow[n]{}\bullet\ \cdots\ \bullet\xleftarrow[n]{}\bullet\ \cdots\ \bullet\xrightarrow[n]{}\bullet\ \cdots\ \bullet\xleftarrow[n]{}\bullet\ \cdots \quad \text{etc}$$

with variables

$$\underline{x}_1 = \left(\frac{1+y^n}{x}, y\right)$$

$$\underline{x} = (x, y)$$

$$\underline{x}_2 = \left(x, \frac{1+x^n}{y}\right)$$

$$\underline{x}_{21} = \left(\frac{1 + \left(\frac{1+x^n}{y}\right)^n}{x}, \frac{1+x^n}{y}\right) \quad \text{etc.}$$

But

$$\frac{1 + \left(\frac{1+x^n}{y}\right)^n}{x} = \left(\frac{1+x^n}{y}\right)^n \frac{1+y^n}{x} - \frac{(1+x^n)^n - 1}{x}$$

$$\in \mathbb{Q}\left[x, y, \frac{1+x^n}{y}, \frac{1+y^n}{x}\right].$$

So, by induction, ~~all~~ ~~cluster~~ all cluster variables are contained in the polynomial algebra

$$\mathbb{Q}\left[x, y, \frac{1+x^n}{y}, \frac{1+y^n}{x}\right]$$

and so

$$\mathcal{A}(Q) = \mathbb{Q}[x, y, x', y'] \big/ \langle xx' = 1+y^n,\ yy' = 1+x^n \rangle$$

Given a quiver Q with variables $\underline{x}$ (a "seed", written as Σ, say), write the Laurent polynomials in $\underline{x}$ as

$$\mathbb{Z}_{\underline{x}} = \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

Then

$$\mathcal{U}(Q, \underline{x}) = \mathbb{Z}_{\underline{x}} \cap \bigcap_{i=1}^{n} \mathbb{Z}_{\underline{x}_i}$$

is called the upper bound of $Q, \underline{x}$.

<u>Easy thm</u> If Σ' is obtained from Σ by mutation, $\mathcal{U}(\Sigma') = \mathcal{U}(\Sigma)$.

<u>Idea</u> Suffices to check when Σ' obtained from Σ by mutating one variable x in Σ.

To illustrate the computation, suppose x has one neighbour, y, and suppose that

$$\Sigma : \underset{x}{\bullet} \longrightarrow \underset{y}{\bullet}$$

Then we have

$$\begin{array}{lll}
\Sigma_y & x \longleftarrow y' & y' = \dfrac{1+x}{y} \\
\uparrow & & \\
\Sigma & \bullet \longrightarrow \bullet & \\
\downarrow & & \\
\Sigma_x & x' \longleftarrow \bullet & x' = \dfrac{1+y}{x} \\
\downarrow & & \\
\Sigma_{xy} & x' \longrightarrow y'' & y'' = \dfrac{1+x+y}{xy}
\end{array}$$

So that

$$\mathcal{U}(\Sigma) = \mathbb{Z}[x^{\pm1}, y^{\pm1}] \cap \mathbb{Z}[x^{\pm1}, y'^{\pm1}] \cap \mathbb{Z}[x'^{\pm1}, y^{\pm1}]$$

$$= \mathbb{Z}[x, y, x', y'] \quad (\text{because } y'^{-1}(1+x) = y \text{ and } x'^{-1}(1+y) = x \text{ etc.})$$

and

$$\mathcal{U}(\Sigma_x) = \mathbb{Z}[x, y, x', y''].$$

But $y'' = x'y' - 1$ and $y' = xy'' - 1$. So $\mathcal{U}(\Sigma) = \mathcal{U}(\Sigma_x)$.

Define

$$\mathcal{L}(\Sigma) = \mathbb{Z}[x_1, x_1', \ldots, x_n'] \subset \mathcal{U}(\Sigma).$$

In the example of $\bullet \longrightarrow \bullet$, above, we saw that

$$\mathcal{L}(\Sigma) = \mathcal{U}(\Sigma).$$

In fact,

<u>hard thm</u> Σ acyclic $\Rightarrow$ $\mathcal{L}(\Sigma) = \mathcal{U}(\Sigma)$.

But, since $\mathcal{A}(\Sigma) = \bigcup_{\tilde{\Sigma}} \mathcal{L}(\tilde{\Sigma})$ over all seed $\tilde{\Sigma}$ and since $\mathcal{L}(\Sigma) \subset \mathcal{A}(\Sigma) \subset \mathcal{U}(\Sigma)$, we have Σ acyclic $\Rightarrow$ $\mathcal{A}(\Sigma) = \mathcal{L}(\Sigma)$. In this case, we can ~~give~~ find an explicit linear basis for $\mathcal{L}(\Sigma)$ given by the monomials $\underline{x}^{(m)}$ where $x_i^{(m_i)} \equiv \begin{cases} x_i^{m_i} & m_i > 0 \\ x_i'^{\,-m_i} & m_i < 0 \end{cases}$.

These clearly span $\mathcal{L}(\Sigma)$ because if an xx' appears in a monomial, we can replace it by $\prod x + \prod x$. But linear independence holds when Σ is acyclic, because then the labels $\{1 \dots n\}$ have a partial ordering from the quiver. This defines a partial order on monomials $\underline{x}^{m}$. This, in turn, defines a partial order on $\underline{x}^{(m)}$ by:

$\underline{x}^{(m)} \prec \underline{x}^{(m')}$ if there is some $\underline{x}^{n}$ in $\underline{x}^{(m)}$ so that $\underline{x}^{n} \prec \underline{x}^{m}$ for all $\underline{x}^{m}$ in $\underline{x}^{(m')}$.

Geometrically, if Σ is acyclic we have

$$A(\Sigma) = \mathcal{L}(\Sigma) = \mathbb{Z}[x_1, \dots, x_n, x_1', \dots, x_n'] / \langle x_1x_1' = \dots, x_2x_2' = \dots, \dots \rangle,$$

so the cluster variety is a total intersection/ A is fin. generated.

A slightly more general class of cluster algebras, that still has this property, is "locally acyclic": which we will define shortly, roughly as "has a finite cover by acyclic varieties."

Let A be cluster algebra in $\mathbb{Q}(x_1 \dots x_n)$. For some cluster variables $y_1 \dots y_m$ call

$$A' = A[y_1^{-1}, \dots, y_m^{-1}]$$

a cluster localization of A if A' is a cluster algebra. E.g. one way to produce such an A' is to take some seed $\Sigma = (Q, \underline{x})$ of A and regard $y_1 \dots y_m$ as frozen, to get a seed Σ'. Then $A' = A(\Sigma')$ is a cluster localization. Examples are below. First:

A is <u>locally acyclic</u> if there exist <u>acyclic</u> cluster localizations A_i such that

- $\bigcap_i A_i = A$
- for every prime ideal $\mathfrak{p} \subset A$ there is an i and a prime ideal $\mathfrak{p}_i \subset A_i$ so that $\mathfrak{p} = \mathfrak{p}_i|_A$.

  (i.e. every point in $\operatorname{Spec} A$ is a point in at least one $\operatorname{Spec} A_i$)

This is an attractive class of cluster algebras b/c

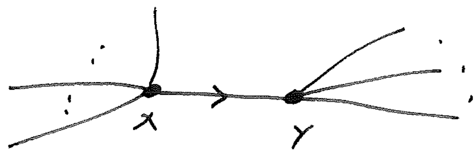
$$A = \bigcap_i A_i = \bigcap_i \mathcal{U}_i \longleftarrow \text{b/c } A_i = \mathcal{U}_i = \mathcal{L}_i \text{ for acyclic}$$

$$= \bigcap_i \mathcal{L}_i .$$

So A is fin generated / Spec A is a complete intersection.

But this wouldn't be useful if ~~[illegible]~~ it weren't possible to easily find acyclic covers.

Consider $A(\Sigma)$ with Σ containing an edge



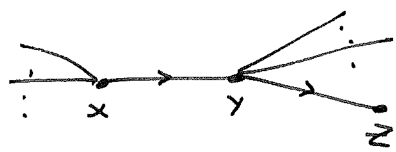
Then

$$A(\Sigma) \subset A(\Sigma_{x \text{ frozen}}) = A[x^{-1}]$$

$$A(\Sigma) \subset A(\Sigma_{y \text{ frozen}}) = A[y^{-1}]$$

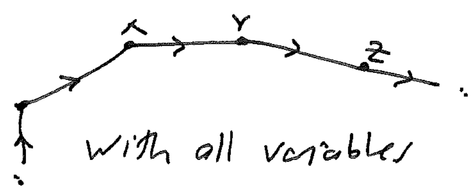
But $A[x^{-1}], A[y^{-1}]$ only form a cover if there is no $\mathfrak{p} \subset A(\Sigma)$ s.t. $x \in \mathfrak{p}$ and $y \in \mathfrak{p}$. So:

Suppose $\exists\, \mathfrak{p} \subset A$ containing x, y $\Rightarrow$ since $yy' = \pi_y^- + \pi_y^+$ and since $x \in \pi_y^-$, we have that $\pi_y^+ \in \mathfrak{p}$ which means there is at least one z to the right of y



such that $z \in \mathfrak{p}$

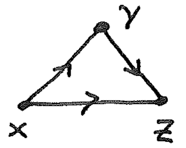
$\Rightarrow$ ...... $\Rightarrow$ can form an infinite path in both directions from $x \to y$,



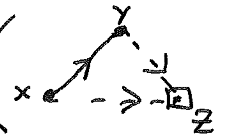
with all variables in $\mathfrak{p}$

So, if there is no such path in Σ, we call

x, y a good pair and $\operatorname{Spec} A[x^{-1}] \cap \operatorname{Spec} A[y^{-1}]$ is a covering.

e.g. $\Sigma =$ 

then we can localize using x, z.

$$A_z = A\Big(\ $$

$$\Big) = \mathbb{Z}[x, y, x', y', z, z^{-1}] \big/ \langle xx' = 1 + yz,\ yy' = x + z \rangle$$

$$A_x = A\Big(\ $$

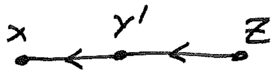
$$\Big) = \mathbb{Z}[x, x^{-1}, y, y', z, z'] \big/ \langle yy' = x + z,\ zz' = xy + 1 \rangle$$

$$A = A_x \cap A_z = \mathbb{Z}[xx'yy'zz'] \big/ \langle xx' = 1 + yz, \dots \rangle$$

In this example, we arrive at an acyclic cover. This is not always possible. Suppose we had

$\Sigma =$ 

Then there are no good pairs to choose. But that doesn't mean $A(\Sigma)$ is not loc. acyclic. Indeed,

$\Sigma_y =$ 

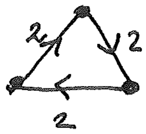
$\qquad y' = \dfrac{x+z}{y}$

so

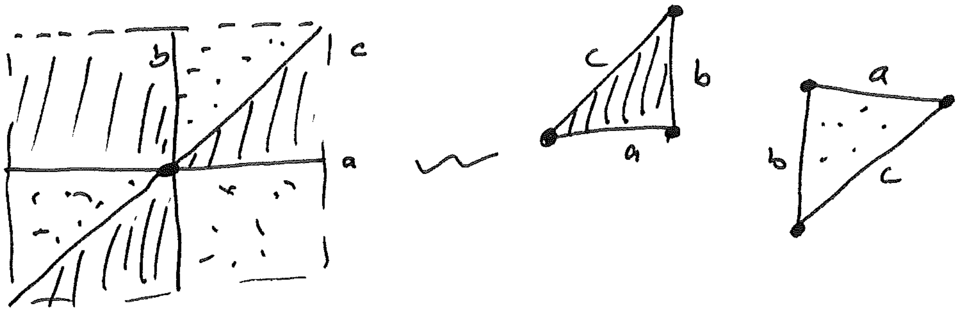
$$A(\Sigma) = A(\Sigma_y) = \mathbb{Z}[x, y', z, x', y'', z'] \big/ \langle xx' = 1 + y',\ y'y'' = x + z,\ zz' = 1 + y' \rangle$$

is acyclic.

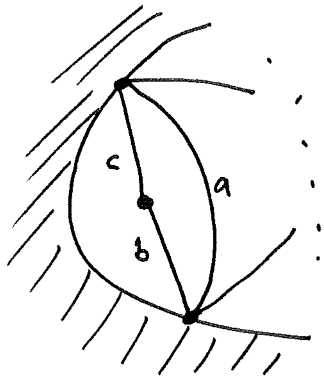
An example which is not loc. acyclic is



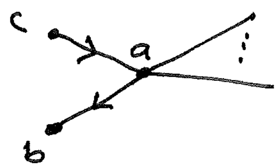
which is the quiver dual to the triangulation



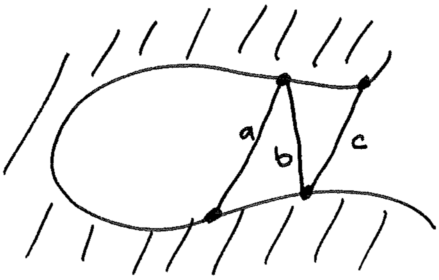
of the punctured torus. Muller has classified those surfaces that do have l.c. acyclic cluster algebras. If a triangulable surface has 2 or more boundary points per bdy component, it is l.c. acyclic. To prove this, make two observations. The triangulation



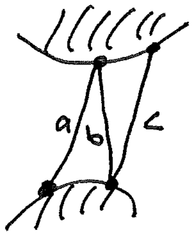
is dual to



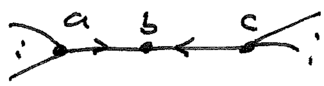
so a, b is clearly a good pair. If a triangulation contains



or



its quiver contains

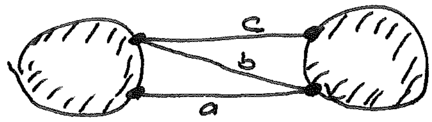


So that a, b is a good pair.

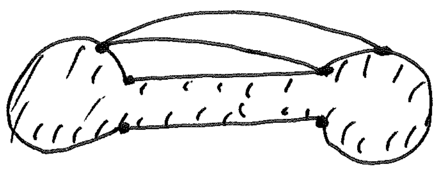
<u>easy thm</u> surface S has ≥ 2 bdy points per component gives a loc. acyclic cluster algebra.

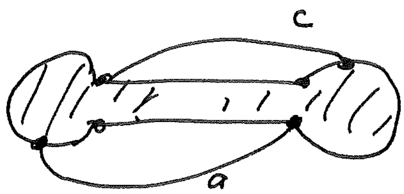
proof. case: S has ≥ 2 bdy components

then take a triangulation that contains

$S =$ 

and freeze $a, b,$ to produce

$S_a =$ 

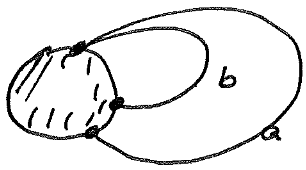
$S_b =$ 

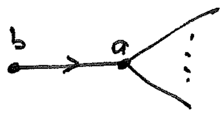
iterate this until

case: S has 1 bdy component

case: $g > 0$

take a triangulation containing



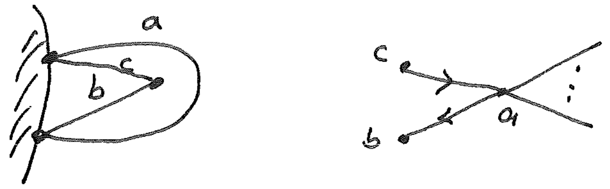


where b is nontrivial & the end point of a may coincide with those of b. a, b is a good pair.

iterate until

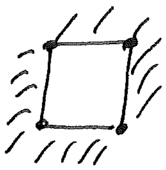
case: $g = 0$

then, if S has a puncture take



and freeze a and b. else, S has no puncture and you can take

where a, b is good, until reduced to



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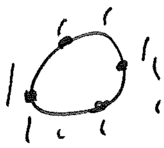
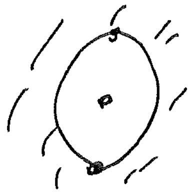
so, it is only necessary to observe that

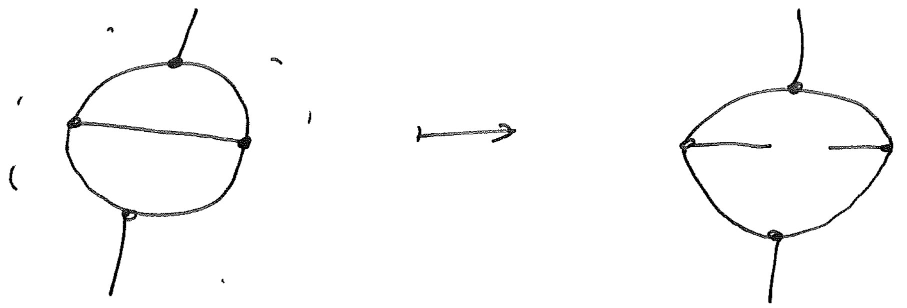


and

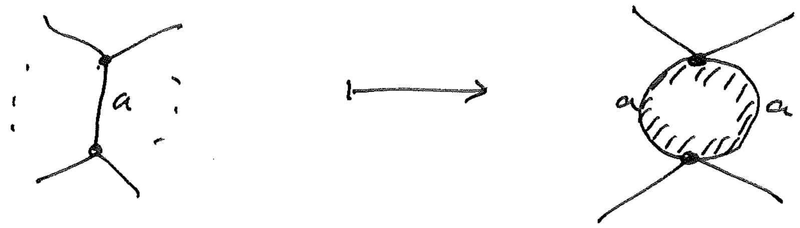


give acyclic quivers, and this concludes the proof that the algorithm works to produce an acyclic cover.

This class of surfaces has an interpretation in terms of the 3-graphs associated to their triangulations. Consider all planar 3-graphs of any # of loops. And consider all the graphs that can be obtained from there by cutting internal edges. This class of graphs is dual to the triangulations of our given class of surfaces (with the usual complication of tagged triangulations leading to $2^{\#\text{loops}}$ triangulations for each tadpole-free graph). Cutting an internal propagator



corresponds at the level of the surface to

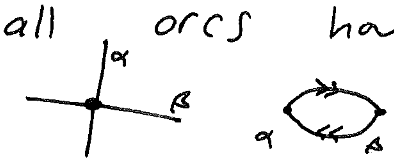


so that all bdy components produced in this way have at least 2 bdy marked points.

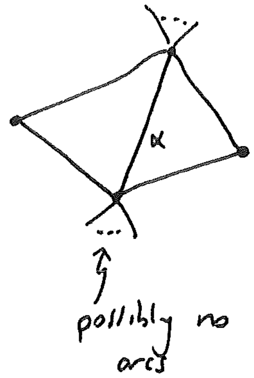
We showed that

≥ 2 marked pts per boundary component $\Rightarrow$ A locally acyclic.

A surface with 1 marked point ~~not~~ does not have any good pairs because all arcs have both incomming and outgoing arrows:



Likewise, a surface without boundary does not have any good pairs. Any arc α looks like



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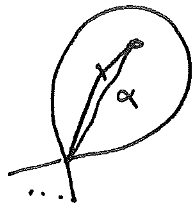
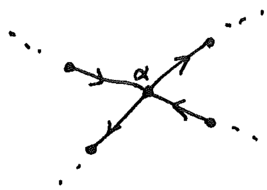


So that, in the quiver,



or

and so no α is a source/sink, so no pair is a good pair.

This leaves: Σ with $\partial\Sigma$ nonempty with ≥ 2 marked points and < 2 marked points on one of the boundary components.