Cartan - Einstein

Black Hole Initiative, Harvard

This talk is about Cartan's work on General Relativity, & his correspondence w̄ Einstein about it. I aim to explain both what Cartan's mathematical contribution was, & some of the historical context that explains why he made his contribution, & why it was of interest to Einstein. To modern eyes, Cartan's work is of interest because it anticipates the idea of a G-bundle, which soon came to play a role in physics with the formulation of gauge theory in the 50s on. I'll start by recalling some of Cartan's work on PDEs by sketching the main theorems.

19^{th} Century PDEs

Cartan was building on & generalizing Frobenius' Thm & the Cauchy–Kobalewski Thm. Consider Frobenius' Thm first, via an example:

Example Try to integrate

$$\frac{\partial u}{\partial x} = F(x,y,u)$$
$$\frac{\partial u}{\partial y} = G(x,y,u)$$

for fncts F & G, with initial condition $u(0,0) = c$. We can integrate

$$\frac{\partial u}{\partial x} = F(x,0,u), \quad u(0,0) = c$$

along $y = 0$ to get $u(x,0)$. Then, along the lines $x = \text{const}$ we can integrate

$$\frac{\partial u}{\partial y} = G(x,y,u), \quad u(x,0) = \text{given}.$$

We thereby obtain $u(x,y)$. The resulting $u(x,y)$ might not satisfy

$$\frac{\partial u}{\partial x} = F(x,y,u)$$

away from $y=0$. The condition that this equation holds for all y is

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} - F(x,y,u)\right) \overset{!}{=} 0 .$$

Expanding this,

$$0 \overset{!}{=} \frac{\partial}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial F}{\partial y} - \frac{\partial F}{\partial u}\frac{\partial u}{\partial y}$$

$$\cancel{\epsilon \, \frac{\partial}{\partial x}(a) \; 2 \; a}$$

$$= \frac{\partial}{\partial x} G(x,y,u(x,y)) - \frac{\partial F}{\partial y} - G\frac{\partial F}{\partial u}$$

$$= \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} + F\frac{\partial G}{\partial u} - G\frac{\partial F}{\partial u} .$$

If F & G satisfy this condition, we can integrate the way we want to to find $u(x,y)$.

This is a special case of Frobenius's Thm, which we state in Cartan's notation rather than Frobenius':

<u>Frobenius</u> (∼1817) Let $\alpha^1, \ldots, \alpha^k$ be 1-forms on a mfld M, generating ideal I of p-forms. Locally, $\exists$ coords $y^1, \ldots, y^k$ such that $\alpha^i = dy^i$ iff $dI \subset I$.

dI is the ideal of 2-forms generated by α^i and $d\alpha^i$.

In our example we had

$$\alpha = du - F\,dx - G\,dy.$$

So

$$\begin{aligned} d\alpha &= \partial_y F\, dx\,dy - \partial_x G\, dx\,dy \\ &\quad + \partial_u F\, dx\,du - \partial_u G\, dy\,du \\ &= \partial_u F\, dx \wedge \alpha - \partial_u G\, dy \wedge \alpha \\ &\quad + (\partial_y F - G\partial_u F - \partial_x G + F\partial_u G)\, dx \wedge dy. \end{aligned}$$

We see that $d\alpha$ is in the ideal gen by α iff

$$\partial_y F - G\partial_u F - \partial_x G + F\partial_u G = 0.$$

<u>Example</u> For a more complicated example of Frobenius, consider the system

$$\underline{\nabla} \times \underline{u} = \underline{u} + \underline{J},$$

where $\underline{J}$ is given. 19th century physicists know that

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{V} = 0$$

for any vector field, so

$$\underline{\nabla} \cdot (\underline{u} + \underline{J}) = 0$$

is necessary to integrate the equation. This turns out to be the only condition.

The system is given by

$$\alpha = du - *(u+j)$$

where u, j are the 1-forms $\Sigma u^i dx^i$, $\Sigma J^i dx^i$. Frobenius' condition is

$$d\alpha = -d*(u+j) \in I$$

or

$$d*(u+j) = 0 \quad (\text{i.e. } \nabla \cdot (\underline{u} + \underline{J}) = 0).$$

Much later than Frobenius, the Cauchy-Kobolewski Thm deals with more complicated-looking systems of PDEs. It was originally stated something like this:

<u>Cauchy-Kobalewski</u> Let $f(x)$ and $G(x, y, a, b)$ be real-analytic. Then the PDE

$$\frac{\partial F(x,y)}{\partial y} = G(x, y, F, \partial_x F)$$

$$F(x, 0) = f(x)$$

is locally integrable.

Sometimes much more is possible. E.g. the heat equation

$$\partial_t^2 F = \nabla^2 F$$

can be integrated for any <u>smooth</u> initial data.

An example where real-analyticity is important is

$$\partial_t^2 F + \partial_x^2 F + \partial_y^2 F = 0.$$

If we ~~know~~ prescribe

$$F(t,x,y)\Big|_{\partial B} = u(t,x,y)$$

at the boundary of a ball B around $(0,0,0)$ with radius R, then we can use Poisson's kernel

$$P(\underline{x},\underline{y}) = \frac{R^2 - \|\underline{x}\|^2}{A\,R\,\|\underline{x}-\underline{y}\|^3} \qquad \left(A = 2\pi^{3/2}/\Gamma(3/2)\right)$$

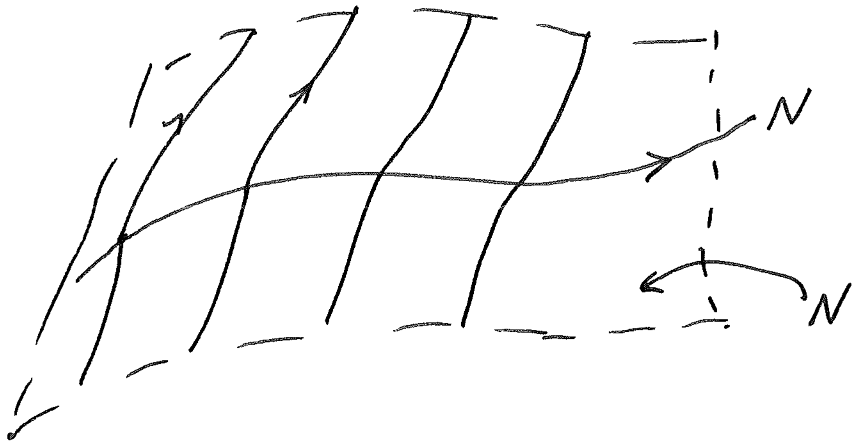
to solve

$$F(\underline{x}) = \int_{\partial B} d^2\underline{y}\; P(\underline{x},\underline{y})\,u(\underline{y}).$$

But P has a power series development in the variables $\underline{x} = (t,x,y)$, so $F(\underline{x})$ (& therefore $u(\underline{y})$) must be real-analytic.

The Cartan-Kähler Thm is a big generalization of Frobenius' Thm, & Cartan had to rely on Cauchy-Kobalewski to state it. The idea is as before: we have an ideal I of p-forms on M, and want to find submanifolds N such that the forms $\varphi \in I$ vanish when restricted to N, $\varphi|_N = 0$ (i.e. the pullback is zero).

Given such an N of dim d, we can try to find N' containing N of dim $d+1$ by integrating.



To check if N is an integral submfld for I is a purely local question:

For all $x \in N$, and $\varphi \in I$, $\varphi|_{T_x N} = 0$.

To state Cartan-Kähler, introduce notation:

$$V_p(I)_x := \left\{ \substack{\text{planes} \\ \text{thru } 0} \ E \subset T_x M \;\middle|\; \dim E = p,\ \varphi|_E = 0 \ \forall \varphi \in I \right\}$$

For any subspace $E \subset T_x M$,

$$H(E)_x := \{ v \in T_x M \mid \varphi(v, e_1, \dots, e_p) = 0 \quad \forall \varphi \in I, \}.$$

$\{e_i\}$ a basis spanning E

Notice that, for a subspace $E' \supset E$,

$$E' \in V_{p+1}(I)_x \quad \text{iff} \quad E' \subset H(E)_x .$$

Write

$$r(E) = \dim H(E)_x - p - 1 .$$

<u>Cartan-Köhler</u>: Suppose N is an integral p-mfld for I. Suppose $r(T_x N)$ is ~~an~~ a non-negative constant, $r \geq 0$.

~~If then~~ If we can choose a codim-r mfld $R \supset N$ such that

$$T_x R \perp H(T_x N) \quad \forall x \in N$$

then $\exists N'$ an integral $p+1$-mfld, with $N \subset N' \subset R$.

Cartan used this to prove that the following isometric embedding problem is locally integrable:

Given n-mfld M with metric g,

find

$$\underline{f} : M \hookrightarrow \mathbb{R}^N, \quad N = n(n+1)/2,$$

such that

$$(*) \qquad g_{ab} = \partial_a \underline{f} \cdot \partial_b \underline{f}$$

(i.e. the induced metric on $\mathrm{Im}\, \underline{f}$ pulls back to g).

Notice that $(*)$ is <u>not</u> a PDE in Cauchy-Kovalevski form, so Cartan's method was needed. More important for our story is that Cartan introduced the Frame bundle to prove $(*)$ is integrable.

<u>Cartan-Janet:</u> $(*)$ is locally integrable.

Sketch: The graph of $\underline{f}$ is a submfld of $M \times \mathbb{R}^N$. We enlarge the problem by looking for a submfld of $M \times \mathcal{F}(\mathbb{R}^N)$, where $\mathcal{F}(\mathbb{R}^N) = \mathbb{R}^N \times \left\{ \begin{matrix}\text{orthonormal sets of} \\ N \text{ basis vectors}\end{matrix} \right\}$ is the frame bundle.

Let η_i be 1-forms on M such that

$$g = (\eta_1)^2 + \dots + (\eta_n)^2$$

and

$$d\eta_i + \sum_j \eta_{ij} \wedge \eta_j = 0$$

for some 1-forms η_{ij}.

Denote a section of $\mathcal{F}(\mathbb{R}^N)$ by 1-forms $\{\omega_i\}$ on $\mathbb{R}^N$, with $d\omega_i + \omega_{ij} \wedge \omega_j = 0$.

Then study the ideal generated by

$$I : \begin{cases} \omega_i - \eta_i & i = 1 \dots n \\ \omega_j & j = n+1 \dots N \\ \omega_{ij} - \eta_{ij} & i,j = 1 \dots n. \end{cases}$$

Since

$$d(\omega_i - \eta_i) = -\sum (\omega_{ij} - \eta_{ij}) \wedge \eta_j \mod I$$
$$= 0 \mod I,$$

it suffices to study

$$I' : \begin{cases} \omega_i - \eta_i & i = 1 \dots n \\ \omega_j & j = n+1 \dots N \end{cases}.$$

Applying Cartan-Kähler to I' gives n-submfld on $M \times \mathcal{F}(\mathbb{R}^N)$.

Get desired result by projecting $\mathcal{F}(\mathbb{R}^N) \to \mathbb{R}^N$.

Generalizing the $\mathcal{F}(\mathbb{R}^N)$ introduced in the proof, Cartan defined the frame bundle of a mfld

$$\begin{array}{c} \mathcal{F}(M) \\ \downarrow \\ M \end{array},$$

sections of which are orthonormal frames $\{e^1, \dots, e^n\}$ of vectors such that $e^i \cdot e^j = \delta^{ij}$.

Dually, we have coframes $\{\omega^1, \dots, \omega^n\}$ for every frame, (i.e. $\langle \omega^i, e_j \rangle = \delta^i_j$).

A connection on $\begin{array}{c} \mathcal{F}(M) \\ \downarrow \\ M \end{array}$ is defined by

$$\nabla e_i = \sum_j e_j \otimes \Gamma_i^{\ j}$$

for some 1-forms $\Gamma_i^{\ j}$. (i.e. $\nabla_V e_i = \sum e_j \langle \Gamma_i^{\ j}, V \rangle$)

The torsion & Riemann tensors can be written in terms of the coframe & $\Gamma_i^{\ j}$ as

$$T^i = d\omega^i + \sum_j \Gamma_j^{\ i} \wedge \omega^j$$

$$R^i_{\ j} = d\Gamma_j^{\ i} + \sum_k \Gamma_k^{\ i} \wedge \Gamma_i^{\ k}$$

They satisfy the Bianchi identities

$$D T^i := dT^i + \sum \Pi^i{}_j \wedge T^j$$

$$= d\Pi^i{}_j \wedge \omega^j - \cancel{\Pi^i{}_j \wedge d\omega^j} + \Pi^i{}_j \wedge \Pi^j{}_k \wedge \omega^k + \cancel{\Pi^i{}_j \wedge d\omega^j}$$

$$= R^i{}_j \wedge \omega^j$$

$$D R^i{}_j := dR^i{}_j + \Pi^i{}_k \wedge R^k{}_j - \Pi^k{}_j \wedge R^i{}_k$$

$$= d\Pi^i{}_k \wedge \Pi^k{}_j - \Pi^i{}_k \wedge d\Pi^k{}_j$$

$$+ \Pi^i{}_k \wedge d\Pi^k{}_j + \Pi^i{}_k \wedge \Pi^k{}_l \wedge \Pi^l{}_j$$

$$- \Pi^k{}_j \wedge d\Pi^i{}_k - \Pi^k{}_j \wedge \Pi^i{}_l \wedge \Pi^l{}_k$$

$$= 0 .$$

Cartan, in a long monograph in 1933, re-wrote much of what was known about GR & Riemannian geometry in terms of the Frame bundle. This was mostly a notational exercise; however, in one respect it wasn't, since Cartan allowed for $T^i \neq 0$ ($T^i{}_{[jk]} \neq 0$ in conventional notation) throughout his discussion. As we will see, Einstein ~~had~~ would be ~~been~~ led to consider $T^i \neq 0$ for quite different reasons, about 5 years later.

Cartan's equations (generalizing Einstein's to $T \neq 0$) follow very directly from the Einstein-Hilbert action,

$$\mathcal{S} = \int \star R.$$

In components,

$$R^i{}_j = R^i{}_{jkl}\, \omega^k \wedge \omega^l$$

so that

$$\begin{aligned}\epsilon_{ijkl}\, \omega^i \wedge \omega^j \wedge R^{kl} &= 2\, \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4\, R^{kl}{}_{kl} \\ &= 2 \star R.\end{aligned}$$

Then

$$\begin{aligned}\delta \mathcal{S} = \int & \epsilon_{ijkl}\, \delta\omega^i \wedge \omega^j \wedge R^{kl} \\ & + \tfrac{1}{2}\, \epsilon_{ijkl}\, \omega^i \wedge \omega^j \wedge \left(d\,\delta\Gamma^{kl} + \delta\Gamma^{ik} \wedge \Gamma^k{}_l - \Gamma^{ik} \wedge \delta\Gamma_{kl} \right) \\ = \int & \delta\omega^i \wedge \left(\epsilon_{ijkl}\, \omega^j \wedge R^{kl} \right) \\ & + \delta\Gamma^{kl} \wedge \left(\epsilon_{ijkl}\, \omega^k \wedge T^l \right).\end{aligned}$$

so that the eqns are

$$E_i = \epsilon_{ijkl} \, w^j R^{kl} = 0$$

$$E_{ij} = \epsilon_{ijkl} \, w^k T^l = 0$$

Cartan does seem to have appreciated that $T \neq 0$ has physical effects. e.g. if u^μ is tangent to geodesics, then $T \neq 0$ modifies the geodesic deviation eqn to become:

$$[\nabla_\mu, \nabla_\nu] u_\lambda = - R_{\beta\lambda\mu\nu} u^\beta + 2 T_{\mu\nu}{}^\beta \nabla_\beta u_\lambda$$

which adds terms to the Raychaudhuri equations. If $\theta = \nabla_\lambda u^\lambda$ is the expansion parameter,

$$\dot{\theta} = \left\{ - \tfrac{2}{3} \theta^2 + R_{\mu\nu} u^\mu u^\nu + \dots \right\} \leftarrow \text{usual stuff}$$

$$+ 2 \nabla^\mu u^\nu T_{\beta\mu\nu} u^\beta$$

So that T can convert the shear & vorticity of u^μ into focussing / defocussing.

Unaware of Cartan's lengthy monograph,