B5 Gravitational Radiation I

Due end HT5

Merton College

1. (Merging Black Holes)
Using the Schwarzschild metric, work out the surface area of the horizon of a Schwarzschild black hole.

If two well-separated Black Holes with surface areas A_1 and A_2 merge, Hawking argues that the new black hole has surface area

$$A_{new} \geqslant A_1 + A_2 .$$

Find an upper bound for the total ~~amount~~ energy of gravitational radiation released when two black holes of mass m_1 and m_2 merge.

2. (Linearised Field Equations)
Einstein's equation is $G_{ab} = 8\pi G T_{ab}$. Now put

$$g_{ab} = \eta_{ab} + h_{ab} \qquad (\text{i.e. "}g\text{ is nearly flat"})$$

where h_{ab} is small. To first order in h_{ab}, show that

$$R_{abcd} = \frac{1}{2}\left(h_{ac,bd} + h_{bd,ac} - h_{ad,bc} - h_{bc,ad}\right).$$

What is Einstein's equation to first order in h_{ab}?

To simplify your answer, write everything in terms of

$$\bar{h}_{ab} = h_{ab} - \tfrac{1}{2}\eta_{ab} h. \qquad (h = \eta^{ab} h_{ab})$$

Show that Einstein's equation to first order in $\bar{h}_{ab}$ becomes

$$\partial^2 \bar{h}_{ab} = 16\pi G\, T_{ab} \qquad (*)$$

if you assume that $\partial^a \bar{h}_{ab} = 0$.

2*. (Optional: Gauge freedom)
Consider a coordinate transformation

$$x'^a = x^a + f^a(x).$$

Working to first order in f^a and h_{ab}, show that

$$h'_{ab} = \partial_a f_b + \partial_b f_a + h_{ab}.$$

Does there exist a unique choice of $f^a(x)$ so that $\partial^a \bar{h}'_{ab} = 0$? How does this bear on Q2?

3. (Gravitational Waves)
Find a gravitational wave solution $h_{ab}(x)$ to the linearised Einstein equation $(*)$ in a vacuum $(T_{ab} = 0)$, that has momentum k^a. (I.E. $\partial_c h_{ab}(x) = i k_c \cdot h_{ab}(x)$.)

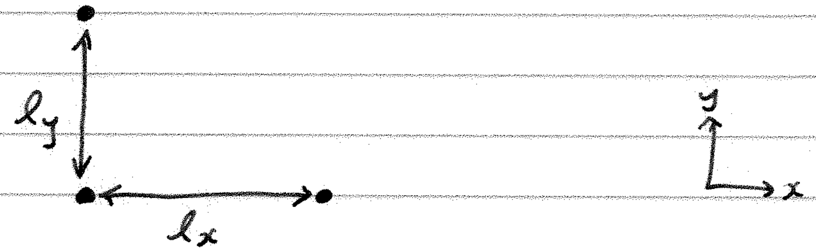
What is the mass of a gravitational wave?

4. (Measuring Gravitational Waves)

Suppose $h_{00} = 0$ and $h_{0i} = 0$ $(i = 1, 2, 3)$. What are the only non-zero Christoffel symbols? (To first order in h_{ab}.)

Show that the worldlines "x^i = constants" are geodesics (to first order in h_{ab}).

Consider three such worldlines at positions $(0,0,0)$ and $(s,0,0)$ and $(0,s,0)$.



A gravitational wave $\bar{h}_{ab} = A_{ab}\, e^{ik\cdot x}$ with

$$k^a = (1,0,0,1) \quad \text{and} \quad A_{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$$

passes through. What are the proper distances l_x and l_y as a function of time?

These oscillations could be measured using interferometry. What would be a good frequency of light to use?

5. (Gravitational Waves Production)

How are gravitational waves produced? Earlier, we found equation (*). Solve this equation.

Suppose T_{ab} is nonzero only near $\underline{x} = 0$. To leading order in $|\underline{x}| = r$, show that

$$\bar{h}_{ij}(t, \underline{x}) = \frac{4G}{r} \int d^3\underline{y} \; T_{ij}(t + |\underline{x} - \underline{y}|, \underline{y}).$$

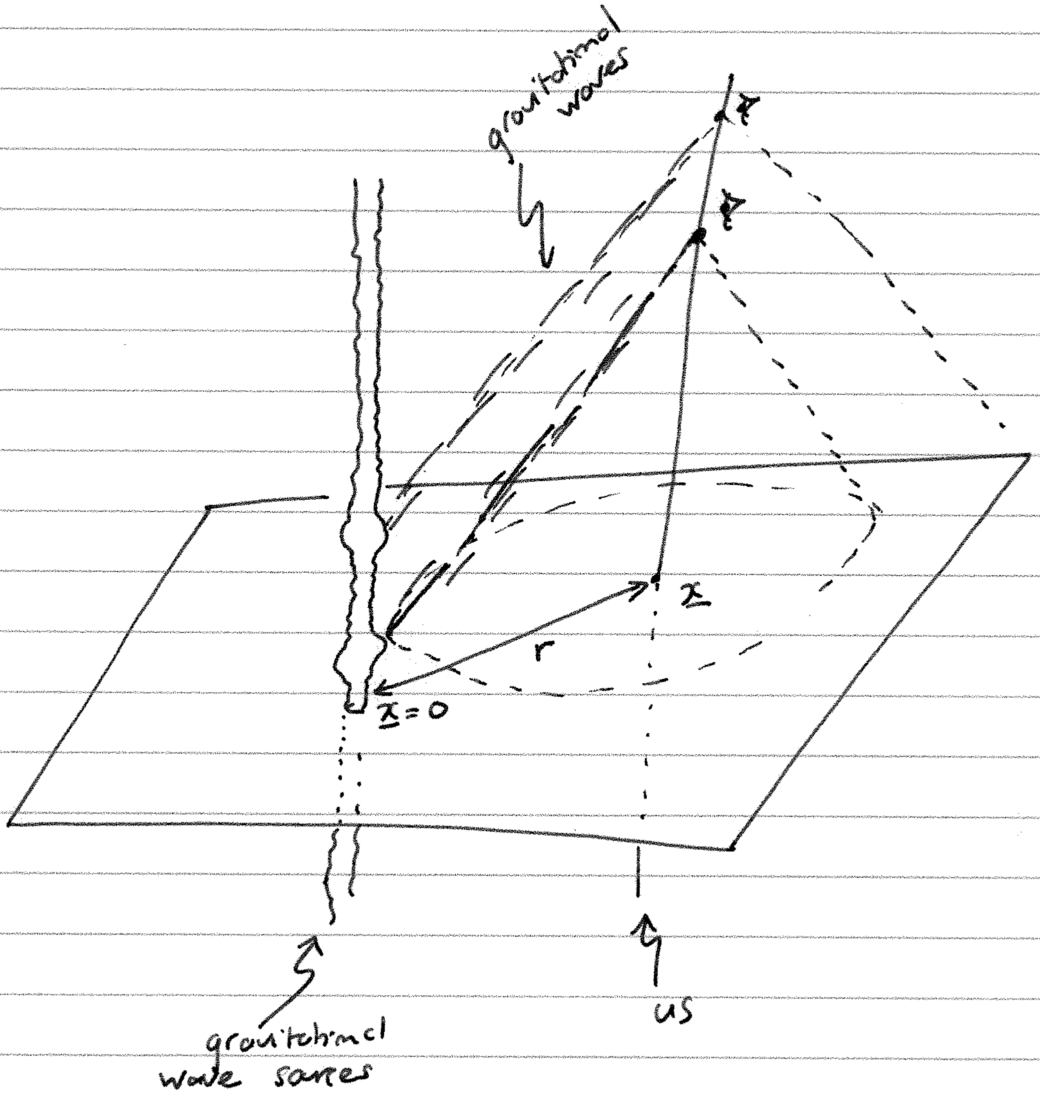
If you want to, prove that

$$2T_{ij} = (T_{00} x^i x^j)_{,00} + \text{total spatial divergence}$$

(by expanding $(T^{ab} x^i x^j)_{,ab}$ out). Infer that

$$\bar{h}_{ij}(t, \underline{x}) = \frac{2G}{r} \ddot{I}_{ij}$$

where I_{ij} is the 2^{nd} moment of T_{00}.



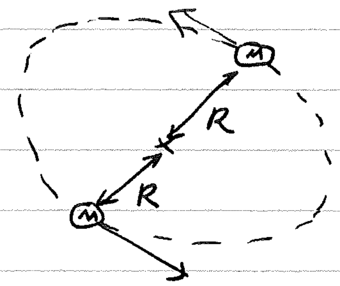
B5 Gravitational Radiation II

Due end HT6

Merton College

1. (Some Newtonian Physics)
State and prove the (Newtonian) virial theorem.

Consider two masses, mass M, seperated by a distance $2R$ and orbiting their centre of mass.



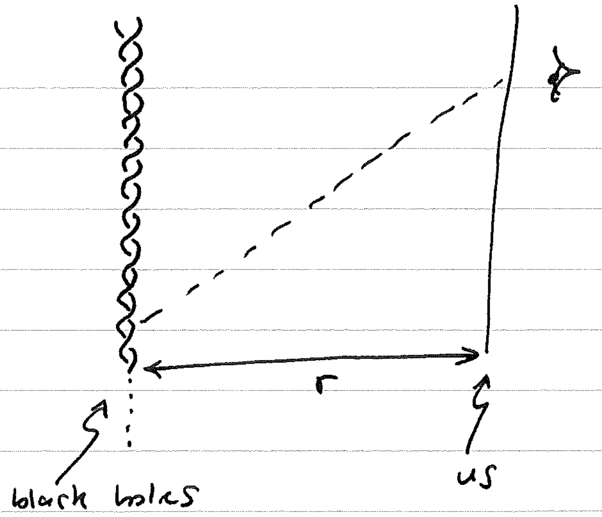
What is the total energy of the system? What is the angular velocity of the masses?

2. (Radiation From Binary System)
Consider two black holes orbiting each other. We approximate their motion using Newtonian physics (Q1). So, in (t,x,y,z) coordinates,

$$T_{00} = M\,\delta(z)\Big[\delta(x - R\cos\omega t)\,\delta(y - R\sin\omega t) + \delta(x + R\cos\omega t)\,\delta(y + R\sin\omega t)\Big].$$

We observe the binary from a distance r. We experience a metric $g = \eta + h$, where h is small. What is h_{ij} to leading order in $1/r$?

[Hint: use HT5, Q5.]



What is the frequency of the waves we observe? Show that the amplitude of the waves we observe satisfies

$$\text{ampl} \propto \frac{G}{r} E_{tot}$$

where E_{tot} is the total (Newtonian) energy of the binary.

3. (Energy Lost By Binary System)
In units where $c=1$, show that the binary system radiates with power

$$P = -\frac{2G^4M^5}{5R^5}.$$

Restore c to this equation.

[Hint: you may use $P = -\frac{1}{5} G \cdot \dddot{J}_{ij} \dddot{J}^{ij}$, where $J_{ij} = I_{ij} - \frac{1}{3}\delta_{ij} I_{kk}$ and dots denote derivatives with respect to t.]